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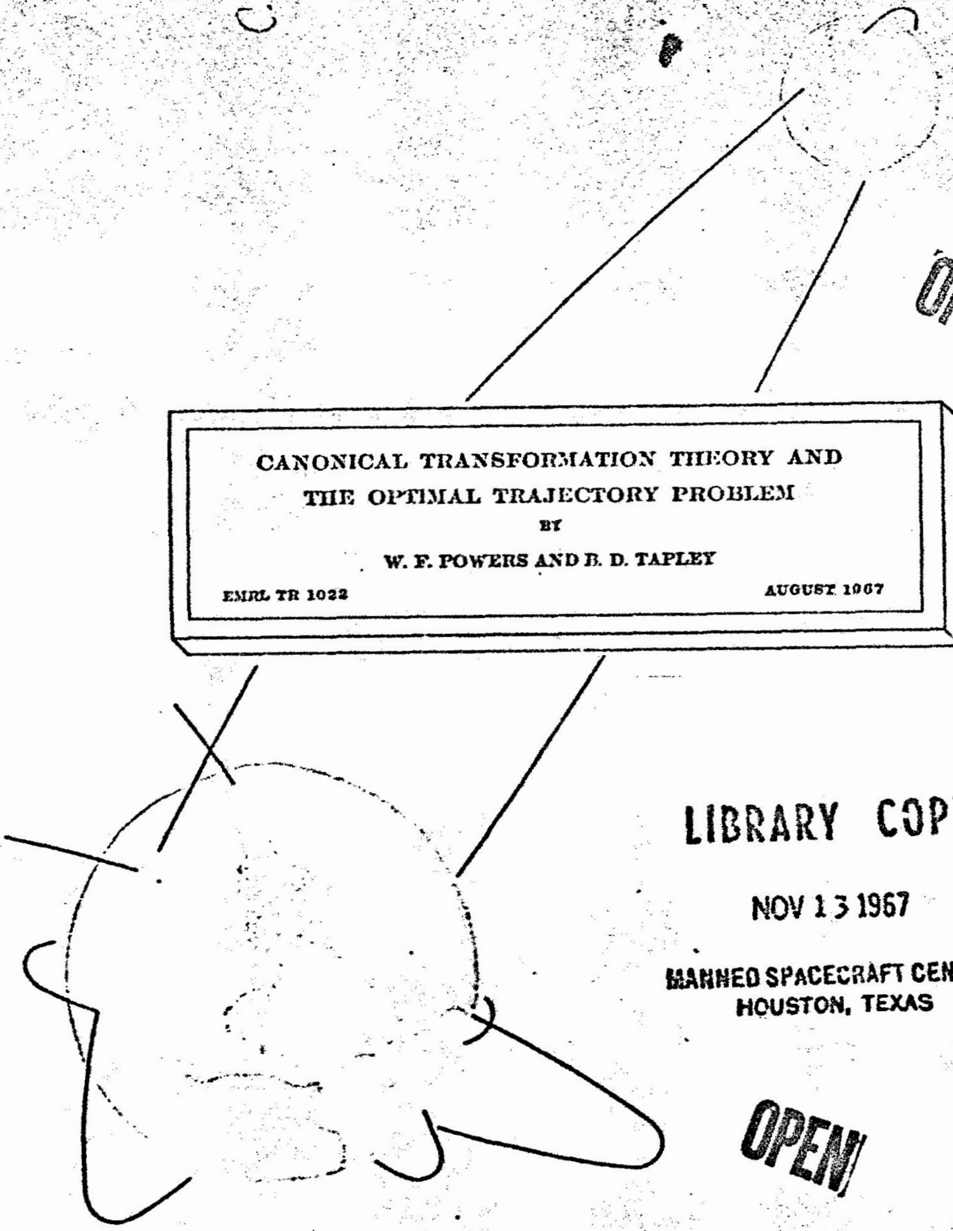

CANONICAL TRANSFORMATION THEORY AND THE OPTIMAL TRAJECTORY PROBLEM

BY

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ABSTRACT

This investigation is concerned with the application of canonical transformation theory to the optimal trajectory problem. For a large class of trajectory optimization problems, the control variables can be uniquely determined as functions of the state variables and Lagrange multipliers. In this case, a new Hamiltonian, which only depends upon state variables (generalized coordinates) and Lagrange multipliers (generalized momenta), can be defined from the generalized variational Hamiltonian. Then, the classical perturbation theories of Hamiltonian mechanics are suitable for the optimal trajectory problem.

In this study, the basic elements of canonical transformation theory are developed with consideration to the optimal trajectory problem. The theory is used to obtain solutions to the Hamilton-Jacobi equation for the coast-arc problem (i.e., the optimal trajectory problem when thrust is zero) in both polar and spherical coordinates. These solutions may be used as base solutions for a canonical perturbation analysis or for the determination of qualitative aspects of the optimal low-thrust problem.

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BASIC ELEMENTS OF CANONICAL TRANSFORMATION THEORY

I.1. Introduction

In recent years considerable work has been expended on the problems of trajectory analysis and guidance theory for continuously-thrusting space vehicles. The main problem in these areas is the determination of the solution to a vector two-point boundary value problem. Since the usual problem is highly nonlinear most of the solutions have been numerical. While the numerical solutions are valuable, analytic solutions are needed for a complete understanding of the problem. Hence, the need exists for approximate analytic solutions to such mission oriented problems as interplanetary transfers, planetary escape trajectories, etc. It is anticipated that the abundance of available numerical solutions can be used as a guide in the determination of the desired analytic solutions.

Based primarily on the work of Poincare[1*], investigators in celestial mechanics and atomic physics have obtained both qualitative characteristics and approximate analytic solutions to many nonlinear problems by application of various canonical perturbation theories. Since the differential equations which describe the optimal trajectory problem can be represented as a Hamiltonian system of differential equations (i.e., there exists a function $H(x, \lambda, t)$ such that the equations $\dot{x}_i = \frac{\partial H}{\partial \lambda_i}$, $\dot{\lambda}_i = -\frac{\partial H}{\partial x_i}$, for $i = 1, \ldots, n$, are the differential equations which define the optimal motion of some physical

*Numbers refer to the listings in the Reference section.

process), it is reasonable to assume that Hamiltonian perturbation theory may give new analytic information for various classes of problems.

In Part I of this report, the basic theory of Hamiltonian systems required in trajectory analysis and guidance theory is presented. The treatment will proceed from the general concepts to the theory most used in applications. The theory has largely been drawn from Wintner[2], Siegel[3], Goldstein[4], Gelfand and Fomin[5], and Born[6].

In Part II, solutions to generalized Hamilton-Jacobi equations when thrust is zero are obtained in both polar and spherical coordinates. These solutions, which can be used as closed-form solutions to the inverse square gravitational field coast-arc problem, are then treated as base solutions for a canonical perturbation theory. In the planar case, the perturbation equations are developed for a continuously thrusting vehicle.

Before turning to the discussion of the basic elements of canonical transformation theory, a few remarks on the notation used in the subsequent developments are in order.

(1) Matrices which are not vectors will be denoted by capital Arabic letters, e.g., $A, M, N, J,$ etc.

(2) The transpose of a matrix, say A, will be denoted by A^T, the inverse by A^{-1}.

(3) Vectors will be treated as $n \times 1$ matrices (i.e., as column vectors), e.g.,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

(4) Let $H(x_1, \ldots, x_n)$ be a scalar function. Then, the gradient of H with respect to the vector x will be represented as a column vector, i.e.,

$$\frac{\partial H}{\partial x} = \begin{bmatrix} \frac{\partial H}{\partial x_1} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial H}{\partial x_n} \end{bmatrix}$$

I.2 Mathematical Preliminaries

In this section some basic concepts and theorems from algebra and analysis needed in the subsequent discussion will be presented for easy reference.

I.2A The Implicit Function Theorem

The majority of the transformations encountered in Hamiltonian systems are nonlinear, so it is necessary to know the points in the region of interest at which the transformations are defined. Furthermore, since implicit functional forms will be encountered, frequent use will be made of the following well known theorem.

Implicit Function Theorem[7]: Let $f_i(x_1, \ldots, x_n, y_1, \ldots, y_m)$ $(i = 1, \ldots, n)$ be n functions such that there exists a point $(x_1^o, x_2^o, \ldots, x_n^o, y_1^o, \ldots, y_m^o)$, in the domain of the definition of the f_i, where $f_i(x_1^o, \ldots, x_n^o, y_1^o, \ldots, y_m^o) = 0$ for each $i = 1, \ldots, n$. Further, assume that each of the f_i are of class C^1(i.e., the f_i and their partial derivatives with respect to each of their arguments are

continuous) and that $\det[\frac{\partial f_i}{\partial x_j}] \neq 0$ at the point $(x_1^o, \ldots, x_n^o, y_1^o, \ldots, y_m^o)$.
Then,

(i) there exists in a neighborhood of the point $(y_1^o, \ldots, y_m^o)$ a unique system of continuous functions $x_i = \phi_i(y_1, \ldots, y_m)(i = 1, \ldots, n)$ which satisfies both $x_i^o = \phi_i(y_1^o, \ldots, y_m^o)$ and $f_i[\phi(y^o), y^o] = 0$ $(i = 1, \ldots, n)$; and

(ii) the partial derivatives $\frac{\partial \phi_i}{\partial y_j}$ exist in some region, are continuous functions of $(y_1, \ldots, y_m)$ in this region, and can be found by solving the equations:

$$\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} \frac{\partial \phi_j}{\partial y_k} + \frac{\partial f_i}{\partial y_k} = 0 . \qquad (i = 1, \ldots, n; \ k = 1, \ldots, m)$$

I. 2B First-Order Partial Differential Equations

This report is mainly concerned with one partial differential equation, i.e., the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H(x, \frac{\partial S}{\partial x}, t) = 0.$$

Since the dependent variable S appears only as a derivative in this equation, the forthcoming analysis will likewise be concerned with first-order equations which contain the dependent variable only through its derivatives. In the following discussion, two methods for the solution of first-order partial differential equations are presented: the separation of variables method and the method of characteristics. Intimately related to these methods is the theory of Pfaffian systems, which is also discussed. More complete descriptions of the methods and theory can be found in References 3, 9, and 10.

Consider a general first-order partial differential equation of the form:

$$F(x_0, x_1, \ldots, x_n, \frac{\partial S}{\partial x_0}, \frac{\partial S}{\partial x_1}, \ldots, \frac{\partial S}{\partial x_n}) = 0. \qquad (1)$$

It will be convenient to introduce the following definition:

$$p_i \equiv \frac{\partial S}{\partial x_i}. \qquad (i = 0, 1, 2, \ldots, n)$$

Then Equation (1) becomes

$$F(x_0, x_1, \ldots, x_n, p_0, p_1, \ldots, p_n) = 0. \qquad (2)$$

Definition I.2.1: A differentiable function $S^*(x_0, \ldots, x_n, a_1, \ldots, a_m)$, where the set $\{a_1, \ldots, a_m\}$ consists of $m \leq n+1$ independent parameters, is called a solution of Equation (2) if

$$F(x_0, \ldots, x_n, \frac{\partial S^*}{\partial x_0}, \ldots, \frac{\partial S^*}{\partial x_n}) \equiv 0.$$

Definition I.2.2: A solution

$$S^* = S(x_0, \ldots, x_n, a_1, \ldots, a_n) + A$$

of Equation (2) which depends on $(n+1)$ independent parameters $\{a_1, \ldots, a_n, A\}$ is called a complete solution if the matrix $[\frac{\partial^2 S}{\partial x_i \partial a_j}]$, with $i = 0, \ldots, n$; $j = 1, \ldots, n$, has rank n.

Since the dependent variable S enters Equation (2) only through its derivatives, the additive constant A in the above definition has no effect on Equation (2). Therefore, this constant will be neglected in further discussions and it will be said that a complete solution of Equation (2) depends upon n independent constants, i.e., the constants $\{a_1, \ldots, a_n\}$.

Separation of Variables

The method of separation of variables is most useful as an "inspection" method although necessary conditions for an equation to be separable can be developed (e.g., see Reference 11). The inspection properties will be discussed rather than the formal development of the necessary conditions for separability since one usually employs the method of characteristics if a separation of variables is not possible.

Suppose that, by inspection, Equation (2) can be written in two parts so that one of the parts contains, at most, one of the independent variables, say x_j, and its associated partial derivative p_j :

$$F_1(x_j, p_j) - F_2(\tilde{x}, \tilde{p}) = 0,$$

where $\tilde{x}$ and $\tilde{p}$ are n-vectors which do not contain x_j and p_j. Then,

$$F_1(x_j, p_j) = F_2(\tilde{x}, \tilde{p}). \qquad (3)$$

Since Equation (3) must hold for all values of $(x_0, x_1, \ldots, x_n)$ in the domain of definition, assume that there exists a solution of Equation (2) of the form:

$$S = S_1(x_j) + S_2(\tilde{x}). \qquad (4)$$

Then, $p_j \equiv \frac{\partial S}{\partial x_j} = \frac{\partial S_1(x_j)}{\partial x_j}$, so p_j is a function of x_j alone. Therefore, with the assumed solution (4), Equation (3) becomes:

$$F_1(x_j, p(x_j)) = F_2(\tilde{x}, \tilde{p}(\tilde{x})). \qquad (5)$$

But $(x_0, \ldots, x_n)$ constitutes a set of $n+1$ independent variables, so an arbitrary variation in any one of the variables (in particular, x_j) does not affect the other variables. Then,

$$F_1(x_j + \delta x_j,\ p_j(x_j + \delta x_j)) = F_2(\tilde{x}, \tilde{p}(\tilde{x}))$$

This condition implies

$$F_1(x_j, p_j) = F_1(x_j + \delta x_j,\ p_j(x_j + \delta x_j)) = \text{constant} \equiv A_1 .$$

Thus, the equation $F_1(x_j, p_j) = A_1$ can be used to solve for

$$p_j = p_j(x_j, A_1) = \frac{\partial S_1}{\partial x_j} .$$

It then follows that

$$S = \int p_j(x_j, A_1)dx_j + S_2(\tilde{x}) .$$

The same procedure may then be applicable to F_2, i.e., there may exist an $x_k \neq x_j$ such that

$$F_3(x_k, p_k) + F_4(\tilde{\tilde{x}}, \tilde{\tilde{p}}) - A_1 = 0 ,$$

where $F_3 + F_4 \equiv F_2$ and the $(n-1)$-vectors $\tilde{\tilde{x}}, \tilde{\tilde{p}}$ do not contain x_j or x_k. In fact, the procedure may be applicable n times, in which case the n constants necessary for a complete solution of Equation (2) will then be defined and the determination of the solution is then simply a matter of integrating terms of the form

$$\int p_i(x_i, A_1, \ldots, A_m)dx_i . \qquad (m \leq n)$$

Thereby, the integration of the partial differential equation represented in Equation (2) has been reduced to quadrature. Even if all n constants for the complete solution cannot be obtained by separation of variables, the attempt should be made to obtain at least a partial separation of variables and then apply the method of characteristics.

Method of Characteristics

Every first-order partial differential equation can be represented by a system of ordinary differential equations which is called the characteristic system for the partial differential equation. The characteristic system for Equation (2) is

$$\begin{aligned} \frac{dx_i}{d\tau} &= \frac{\partial F}{\partial p_i} \\ \frac{dp_i}{d\tau} &= -\frac{\partial F}{\partial x_i} \qquad (i = 0, \ldots, n) \qquad (6) \\ \frac{dS}{d\tau} &= \sum_{i=0}^{n} p_i \frac{dx_i}{d\tau} . \end{aligned}$$

Note that if one lets $\{\tau, F, x_i, p_i\} \equiv \{t, H, x_i, \lambda_i\}$ in the above equations, then the first two sets of equations become Hamilton's equations. Thus, in a dynamical system, Hamilton's equations are the characteristics for the Hamilton-Jacobi equation.

In solving Equation (2), the method of characteristics is used most effectively in conjuction with the method of separation of variables. That is, one first determines as many constants of the complete solution as possible by separation of variables (i.e., a partial separation of variables), say

$A_1, \ldots, A_k$ where $k < n$, so then Equation (2) can be written as

$$F^*(x_k, \ldots, x_n, p_k, \ldots, p_n, A_1, \ldots, A_k) = 0. \tag{7}$$

Note that it is assumed, without loss of generality, that $\{x_0, \ldots, x_{k-1}, p_0, \ldots, p_{k-1}\}$ have been eliminated from the partial differential equation by substitution of the constants $\{A_1, \ldots, A_k\}$. Equation (7) is then a partial differential equation in $n-k+1$ variables instead of $n+1$ variables, as is Equation (2).

The characteristic system for Equation (7) is

$$\begin{aligned} \frac{dx_i}{d\tau} &= \frac{\partial F^*}{\partial p_i} \\ \frac{dp_i}{d\tau} &= -\frac{\partial F^*}{\partial x_i} \qquad (i = k, \ldots, n) \\ \frac{dS^*}{d\tau} &= \sum_{i=k}^{n} p_i \frac{dx_i}{d\tau}, \end{aligned} \tag{8}$$

where S^* is defined by the equation

$$S = \sum_{j=0}^{k-1} \int p_j(x_j)dx_j + S^*(x_k, \ldots, x_n). \tag{9}$$

Then only $n-k$ constant relationships must be found in the first two sets of Equations (8) in order to have the necessary number of constants for a complete solution. If it is not possible to obtain the necessary $n-k$ constant relationships from the characteristic system, then one should find as many as possible and then go back to the separation of variables method and so on until n constants are obtained.

As will be shown later, one of the most powerful aspects of the Hamilton-Jacobi theory is that only n constants of the motion must be found by integration, whereas the solution of Hamilton's equations requires $2n$ integrations. Thus, when the complete solution to the Hamilton-Jacobi equation has been determined, the remaining n constants of the motion can be simply obtained by differentiation.

Pfaffian Systems

If some of the constants for the complete solution of Equation (2) are found by application of the method of characteristics, it is likely that some of the partial derivatives, $\frac{\partial S}{\partial x_i}$, will be of the functional form

$$\frac{\partial S}{\partial x_i} = \frac{\partial S}{\partial x_i}(x_p, x_q, \ldots, x_r),$$

where $\frac{\partial S}{\partial x_i}$ may or may not depend on x_i. Consider, for example, the solution form of Equation (9). The total differential of S is

$$dS = \sum_{j=0}^{k-1} p_j(x_j)dx_j + \sum_{m=k}^{n} \frac{\partial S^*}{\partial x_m}(x_k, \ldots, x_n)dx_m. \tag{10}$$

Thus, to determine the solution S, Equation (10) must be integrated. The integration of the first summation of terms is straightforward, but the integration of the second summation of terms is not since the coefficients of the dx_m may be functions of other variables than x_m. This integration problem has been investigated in the literature and is usually called the Pfaffian problem.

Definition I.2.3[9]: The expression $\sum_{i=1}^{n} G_i(x_1, \ldots, x_n)dx_i$ is called a Pfaffian differential form in n variables; the differential equation

$$\sum_{i=1}^{n} G_i(x_1, \ldots, x_n)dx_i = 0$$

is called the Pfaff differential equation.

With respect to Equation (10) and the above definition, it follows that a Pfaff differential equation of the following form must be integrated

$$dS^* - \sum_{m=k}^{n} R_m(x_k, \ldots, x_n)dx_m = 0, \tag{11}$$

where $R_m(x_k, \ldots, x_n) \equiv \frac{\partial S^*}{\partial x_m}$. In order to determine the generating function S, the solution to Equation (11) must be obtained. For cases where Equation (11) depends on more than two independent variables (including S^*), there does not exist a general integration theory[9]. However, integrals can sometimes be found by inspection of the functional form of the equation. An important example is the following: suppose Equation (11) contains a term of the form $R_m(\tilde{x})dx_m$, where $\tilde{x}$ does not contain x_m. Then S^* must be of the form

$$S^* = R_m(\tilde{x})x_m + S'(\tilde{x}).$$

This can be shown by contradiction, as follows. Assume x_m does not appear linearly in S^*. Then there exist two possibilities:

(i) x_m does not appear explicitly in S^*. This case cannot be true since it would require

$$\frac{\partial S^*}{\partial x_m} \equiv 0 \rightarrow R_m \equiv 0.$$

(ii) x_m appears nonlinearly in S^*. If this is the case, then $\frac{\partial S^*}{\partial x_m}$ must contain x_m explicitly. But,

$$\frac{\partial S^*}{\partial x_m} = R_m,$$

so then R_m is also a function of x_m. But, by hypothesis, this is not true. Thus, S^* can contain x_m only in the product form $R_m x_m$.

I.2C Groups

As will be shown later, the class of canonical transformations can be represented as a group. The definition and some simple properties of a group will now be presented. For a more complete presentation, see References 12 and 13.

Definition .2.4: Let G be a nonempty set and "." an operation defined on G. The set G is called a group with respect to "." if

(i) for each $a, b \in G$, $a \cdot b \in G$; (closure)

(ii) for each $a, b, c \in G$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$; (associativity)

(iii) for each $a \in G$ there exists $e \in G$ such that
$a \cdot e = e \cdot a = a$; (identity)

(iv) for each $a \in G$ there exists $x \in G$ such that
$a \cdot x = x \cdot a = e$. (inverse)

Properties of Groups

(G.1) The identity, e, of a group is unique.

(G.2) For each $a \in G$ there exists a unique inverse $a^{-1} \in G$.

(G.3) If $a, b, c \in G$ and $a \cdot b = a \cdot c$, then $b = c$.

(G.4) If $a, b, c \in G$ and $b \cdot a = c \cdot a$, then $b = c$.

(G. 5) If $a, b \in G$, then there exist elements $p, q \in G$ such that $a \cdot p = b$ and $q \cdot a = b$. In fact, $p = a^{-1} \cdot b$ and $q = b \cdot a^{-1}$.

(G. 6) If $a, b \in G$, then $(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$.

Definition I. 2. 5: A nonempty subset K of a group G is called a subgroup of G if:

(i) $a, b \in K \longrightarrow a \cdot b \in K$;

(ii) $a \in K \longrightarrow a^{-1} \in K$.

I. 2D Symplectic Matrices

Before defining the symplectic matrix, the concept of the canonical matrix (and its properties) must be introduced.

Definition I. 2. 6: Let I_n = $n \times n$ identity matrix and O_n = $n \times n$ zero matrix. Then the $2n \times 2n$ matrix

$$J \equiv \left[\begin{array}{c|c} O_n & I_n \\ \hline -I_n & O_n \end{array}\right]$$

is called the canonical matrix.

Properties of the Canonical Matrix

(C. 1) $J^2 = -I$, where $I \equiv I_{2n}$.

Proof: By straightforward multiplication.

(C. 2) J is nonsingular, i. e., $|J| \neq 0$.

Proof: By (C. 1), $J^2 = -I$. Then, since the determinant of a product is the product of the determinants.

$$|J^2| = |J| \cdot |J| = |-I| = (-1)^{2n} = +1.$$

$\therefore \ |J| = \pm 1 \neq 0.$

(C. 3) $J^{-1} = -J.$

Proof: By (C. 1): $J^{-1}(J^2) = J^{-1}(-I) = -J^{-1}$. But, also:

$J^{-1}(J^2) = (J^{-1}J)J = IJ = J.$

$\therefore \ -J^{-1} = J \rightarrow J^{-1} = -J.$

Definition I. 2. 7: Let M be a $2n \times 2n$ matrix. The matrix M is said to be symplectic if

$$M^T JM = \mu J,$$

where μ is a nonzero scalar constant.

(Note: Siegel[2] and most other texts do not include the constant μ in their definition of a symplectic matrix. However, the purposes of this report are best served by using the above definition. Also, Wintner[1] uses the condition $MJM^T = \mu J$ instead of the one given above. But, as Wintner shows on page 26, the two conditions are equivalent.)

Some important properties of symplectic matrices will now be presented. In this development, M will be assumed to be a $2n \times 2n$ matrix and J will be the canonical matrix.

Property I. 2. 1: If M is symplectic, then M^{-1} exists.

Proof: It must be shown that $|M| \neq 0$. By the definition of a symplectic matrix:

$$|M^T JM| = |M^T| \cdot |J| \cdot |M| = \mu |J|$$

or,

$$|M|^2 |J| = \mu |J| .$$

since $|M^T| = |M|$. But, $|J| \neq 0$, so

$$|M|^2 = \mu$$

$$\therefore \quad |M| = \pm\mu \neq 0, \text{ since } \mu \text{ is nonzero.}$$

<u>Property I. 2. 2</u>: The inverse of a symplectic matrix M is given by

$$M^{-1} = -\frac{1}{\mu} JM^TJ.$$

<u>Proof:</u> By definition, $M^TJM = \mu J$ since M is symplectic. Operate on this equation on the right with M^{-1}, so

$$M^TJ = \mu JM^{-1}.$$

Operate on the left with J^{-1}, so

$$J^{-1}M^TJ = \mu M^{-1}.$$

But, $J^{-1} = -J$, so: $M^{-1} = -\frac{1}{\mu} JM^TJ$.

Now it will be shown that the class of symplectic matrices forms a group. The closure property which is a result of this fact is very important in applications.

<u>Property I. 2. 3</u>: Let S be the class of symplectic matrices of order $2n \times 2n$. Then, S is a group with respect to matrix multiplication.

<u>Proof</u>: Since the product of two matrices of order $2n \times 2n$ is a $2n \times 2n$ matrix, one need only show that for each $M_1, M_2 \in S$ that the product M_2M_1 is a symplectic matrix (i. e., to satisfy closure). Since both $M_1, M_2 \in S$, then $M_1^TJM_1 = \mu_1 J$ and $M_2^TJM_2 = \mu_2 J$. Solving for J yields $J = \frac{1}{\mu_2} M_2^TJM_2$; so then

$$M_1^T [\frac{1}{\mu_2} M_2^T J M_2] M_1 = \mu_1 J$$

$$\frac{1}{\mu_2}(M_1^T M_2^T) J(M_2 M_1) = \mu_1 J$$

$$(M_2 M_1)^T J(M_2 M_1) = (\mu_1 \mu_2) J ,$$

so $(M_2 M_1)$ is symplectic.

The associative property of a group is immediately satisfied since matrix multiplication is associative in general.

The identity property is satisfied by $M \equiv I$, i.e.,

$$I^T J I = J \text{ and } MI = IM = M.$$

Finally, consider M^{-1} as the inverse for M (by Property I.2.1, M^{-1} exists). Clearly $MM^{-1} = M^{-1}M = I$, so one need only show that $M^{-1} \epsilon S$, i.e., $(M^{-1})^T J \; M^{-1} = \mu J$. From matrix theory, if $|M| \neq 0$ then $(M^{-1})^T = (M^T)^{-1}$, so

$$(M^{-1})^T J M^{-1} = (M^T)^{-1} J M^{-1}.$$

But M is symplectic, which implies

$$(M^T)^{-1} = (\mu J M^{-1} J^{-1})^{-1} = \frac{1}{\mu} J M J^{-1}.$$

Thus, $(M^{-1})^T J M^{-1} = [\frac{1}{\mu} J M J^{-1}] J M^{-1} = \frac{1}{\mu} J M M^{-1} = \frac{1}{\mu} J$, so M^{-1} is also symplectic.

I.3 Canonical Transformations

The main problem in trajectory analysis and guidance theory is the integration of the equations of motion and the Euler-Lagrange equations. One can equivalently describe the given second-order system of ordinary differential equations by a system of first-order ordinary differential equations in the Hamiltonian form. Throughout this report, the following definition will be used for a Hamiltonian system.

Definition I.3.1: Let x and λ be n-vectors and t be a scalar. The x_i will be called generalized coordinates and the λ_i will be called generalized momenta. If there exists a scalar differentiable function $H(x, \lambda, t)$ such that:

$$\begin{aligned} \dot{x}_i &\equiv \frac{dx_i}{dt} = \frac{\partial H}{\partial \lambda_i} \\ \dot{\lambda}_i &\equiv \frac{d\lambda_i}{dt} = -\frac{\partial H}{\partial x_i} \end{aligned} \qquad (i = 1, \ldots, n) \tag{12}$$

are the differential equations describing a given dynamical process, then the set $\{H, x, \lambda\}$ is called a Hamiltonian system.

(Note: Notationwise, unless stated otherwise, the variables $\{x, q, Q, \beta\}$ will represent generalized coordinates and $\{\lambda, p, P, \alpha\}$ will represent generalized momenta.)

I.3A The Definition and Necessary and Sufficient Condition

Most optimal trajectory problems are not integrable in closed form, so the system of differential equations which define the problem are usually integrated numerically to obtain the solution. However, there exist other alternatives. Suppose $2n$ independent constants of the motion are known for Equations (12). Then the problem is reduced to the solution of $2n$ algebraic equations. The fundamental objective of canonical transformation theory is to transform the given system of Hamilton's equations into another system of Hamiltonian equations which is readily integrable. In the particular case of the Hamilton-Jacobi transformation, the result is the equilibrium solution (i.e., $2n$ constants of the motion).

Definition I. 3. 2: Let $\{x(q, p, t), \lambda(q, p, t)\} \in C^2$ be a transformation which satisfies the conditions of the implicit function theorem. If for every $H(x, \lambda, t)$ there exists a scalar function $K(q, p, t)$ such that

$$\dot{q}_i = \frac{\partial K}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial K}{\partial q_i}, \qquad (i = 1, \ldots, n)$$

then the transformation is said to be canonical.

Note that the word "every" is emphasized in the above definition. The definition does not say that every transformation which preserves Hamiltonian form is canonical, but only those which preserve Hamiltonian form and are independent of the Hamiltonian function. Thus, if a transformation is canonical, it remains so for every choice of the Hamiltonian.

By adopting the above definition for a canonical transformation, the following necessary and sufficient condition for a canonical transformation can be determined.

Theorem I. 3. 1: Let $\{x(q, p, t), \lambda(q, p, t)\}$ be a transformation which satisfies the conditions of the implicit function theorem, and let M be the

Jacobian matrix of the transformation. Then, $\{x(q, p, t), \lambda(q, p, t)\}$ is a canonical transformation if and only if M is symplectic.

Proof: (Sufficiency $\leftarrow$) Consider the time derivatives of the set $\{x, \lambda\}$

$$\dot{x}_i = \sum_{j=1}^{n} \left[\frac{\partial x_i}{\partial q_j} \dot{q}_j + \frac{\partial x_i}{\partial p_j} \dot{p}_j \right] + \frac{\partial x_i}{\partial t}$$

$$(i = 1, \ldots, n)$$

$$\dot{\lambda}_i = \sum_{j=1}^{n} \left[\frac{\partial \lambda_i}{\partial q_j} \dot{q}_j + \frac{\partial \lambda_i}{\partial p_j} \dot{p}_j \right] + \frac{\partial \lambda_i}{\partial t} .$$

Let M be the Jacobian matrix of the transformation, i.e.,

$$M \equiv \left[\begin{array}{c|c} \dfrac{\partial x_i}{\partial q_j} & \dfrac{\partial x_i}{\partial p_j} \\ \hline \dfrac{\partial \lambda_i}{\partial q_j} & \dfrac{\partial \lambda_i}{\partial p_j} \end{array} \right]$$

Then, in matrix form

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = M \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} + \begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial \lambda}{\partial t} \end{bmatrix} . \qquad (13)$$

Since the given transformation satisfies the implicit function theorem, the inverse transformation exists, i.e., $\{q(x, \lambda, t), p(x, \lambda, t)\}$. Thus,

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = N \begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} + \begin{bmatrix} \frac{\partial q}{\partial t} \\ \frac{\partial p}{\partial t} \end{bmatrix} , \qquad (14)$$

where N is the Jacobian matrix for the inverse transformation, i.e.,

$$N \equiv \left[\begin{array}{c|c} \frac{\partial q_i}{\partial x_j} & \frac{\partial q_i}{\partial \lambda_j} \\ \hline \frac{\partial p_i}{\partial x_j} & \frac{\partial p_i}{\partial \lambda_j} \end{array}\right]$$

It will be shown now that $N = M^{-1}$. Substitution of Equation (14) into Equation (13) gives

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = MN \begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} + M \begin{bmatrix} \frac{\partial q}{\partial t} \\ \frac{\partial p}{\partial t} \end{bmatrix} + \begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial \lambda}{\partial t} \end{bmatrix} .$$

This equation must hold for <u>all</u> Hamiltonian functions, in particular those independent of time. Thus,

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = MN \begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} \rightarrow \quad MN = I$$

and

$$\begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial \lambda}{\partial t} \end{bmatrix} = -M \begin{bmatrix} \frac{\partial q}{\partial t} \\ \frac{\partial p}{\partial t} \end{bmatrix} \rightarrow \begin{bmatrix} \frac{\partial q}{\partial t} \\ \frac{\partial p}{\partial t} \end{bmatrix} = -M^{-1} \begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial \lambda}{\partial t} \end{bmatrix} .$$

Since M^{-1} exists (M is symplectic) and since $MN = I$, it follows that $N = M^{-1}$. By Property I.2.2, $N = M^{-1} = -\frac{1}{\mu} JM^TJ$. Hence, Equation (14) can be expressed as

$$\begin{bmatrix} \dot{q} \\ \\ \dot{p} \end{bmatrix} = -\frac{1}{\mu} \; J M^T J \begin{bmatrix} \dot{x} \\ \\ \dot{\lambda} \end{bmatrix} + \begin{bmatrix} \frac{\partial q}{\partial t} \\ \\ \frac{\partial p}{\partial t} \end{bmatrix} .$$

Upon multiplication by J

$$J \begin{bmatrix} \dot{q} - \frac{\partial q}{\partial t} \\ \\ \dot{p} - \frac{\partial p}{\partial t} \end{bmatrix} = -\frac{1}{\mu} \; J^2 M^T J \begin{bmatrix} \dot{x} \\ \\ \dot{\lambda} \end{bmatrix}$$

But, $J \begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \dot{\lambda} \\ -\dot{x} \end{bmatrix}$ and $J^2 = -I$ so

$$\begin{bmatrix} \dot{p} - \frac{\partial p}{\partial t} \\ \\ -(\dot{q} - \frac{\partial q}{\partial t}) \end{bmatrix} = \frac{1}{\mu} \; M^T \begin{bmatrix} \dot{\lambda} \\ \\ -\dot{x} \end{bmatrix} .$$

Since $\{H, x, \lambda\}$ is a Hamiltonian system

$$\begin{bmatrix} \dot{p} - \frac{\partial p}{\partial t} \\ \\ -(\dot{q} - \frac{\partial q}{\partial t}) \end{bmatrix} = -\frac{1}{\mu} \; M^T \begin{bmatrix} \frac{\partial H}{\partial x} \\ \\ \frac{\partial H}{\partial \lambda} \end{bmatrix} . \tag{15}$$

In scalar form Equation (15) becomes

$$\begin{aligned} \dot{p}_i - \frac{\partial p_i}{\partial t} &= -\frac{1}{\mu} \sum_{j=1}^{n} \left[\frac{\partial x_j}{\partial q_i} \frac{\partial H}{\partial x_j} + \frac{\partial \lambda_j}{\partial q_i} \frac{\partial H}{\partial \lambda_j} \right] \\ -(\dot{q}_i - \frac{\partial q_i}{\partial t}) &= -\frac{1}{\mu} \sum_{j=1}^{n} \left[\frac{\partial x_j}{\partial p_i} \frac{\partial H}{\partial x_j} + \frac{\partial \lambda_j}{\partial p_i} \frac{\partial H}{\partial \lambda_j} \right] \end{aligned} \tag{16}$$

Define the function

$$K(q, p, t) \equiv \frac{1}{\mu} H[\, x(q, p, t), \lambda(q, p, t), t\,] + R(q, p, t), \tag{17}$$

where $\frac{1}{\mu}$ and R are called the multiplier and the remainder function of the canonical transformation, respectively. Then,

$$\begin{aligned} \frac{\partial K}{\partial q_i} &= \frac{1}{\mu} \sum_{j=1}^{n} \left[\frac{\partial H}{\partial x_j} \frac{\partial x_j}{\partial q_i} + \frac{\partial H}{\partial \lambda_j} \frac{\partial \lambda_j}{\partial q_i} \right] + \frac{\partial R}{\partial q_i} \\ \frac{\partial K}{\partial p_i} &= \frac{1}{\mu} \sum_{j=1}^{n} \left[\frac{\partial H}{\partial x_j} \frac{\partial x_j}{\partial p_i} + \frac{\partial H}{\partial \lambda_j} \frac{\partial \lambda_j}{\partial p_i} \right] + \frac{\partial R}{\partial p_i} . \end{aligned} \tag{18}$$

Substitution of Equations (18) into Equations (16) gives

$$\begin{aligned} -\dot{p}_i + \frac{\partial p_i}{\partial t} &= \frac{\partial K}{\partial q_i} - \frac{\partial R}{\partial q_i} \\ \dot{q}_i - \frac{\partial q_i}{\partial t} &= \frac{\partial K}{\partial p_i} - \frac{\partial R}{\partial p_i} . \end{aligned} \tag{19}$$

In the definition of K, given by Equation (17), the term $\frac{1}{\mu} H$ is well-defined, but R is not. Thus, for K to be a Hamiltonian function in the $\{q, p\}$-space, it must be shown that there exists a function $R(q, p, t)$ which satisfies the equations

$$-\frac{\partial p_i}{\partial t} = \frac{\partial R}{\partial q_i} , \qquad \frac{\partial q_i}{\partial t} = \frac{\partial R}{\partial p_i} . \tag{20}$$

For then Equations (19) become

$$\begin{aligned} \dot{p}_i &= -\frac{\partial K}{\partial q_i} \\ \dot{q}_i &= \frac{\partial K}{\partial p_i} , \end{aligned} \qquad (i = 1, \ldots, n)$$

which is a Hamiltonian system in the $\{q, p\}$-space.

Thus, to complete the sufficiency proof, it must be shown that there exists a solution $R(q, p, t)$ to Equations (20) (note that the solution need not be unique). In matrix form, Equations (20) can be written as

$$\begin{bmatrix} \dfrac{\partial R}{\partial p} \\ \\ \dfrac{\partial R}{\partial q} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial q}{\partial t} \\ \\ -\dfrac{\partial p}{\partial t} \end{bmatrix} = J \begin{bmatrix} \dfrac{\partial p}{\partial t} \\ \\ \dfrac{\partial q}{\partial t} \end{bmatrix}$$

or

$$-\begin{bmatrix} \dfrac{\partial R}{\partial q} \\ \\ \dfrac{\partial R}{\partial p} \end{bmatrix} = J \begin{bmatrix} \dfrac{\partial q}{\partial t} \\ \\ \dfrac{\partial p}{\partial t} \end{bmatrix} .$$

Note that the left-hand side of this equation is a gradient. Thus, a solution of Equations (20) exists if the vector

$$J \begin{bmatrix} \dfrac{\partial q}{\partial t} \\ \\ \dfrac{\partial p}{\partial t} \end{bmatrix}$$

is a gradient. Since $\{x(q, p, t), \lambda(q, p, t)\}$ is assumed to be of class C^2, then the functions $\{\dfrac{\partial q_1}{\partial t}, \dots, \dfrac{\partial q_n}{\partial t}, \dfrac{\partial p_1}{\partial t}, \dots, \dfrac{\partial p_n}{\partial t}\}$ are of class C^1. Thus, the functions $\{\dfrac{\partial R}{\partial q_1}, \dots, \dfrac{\partial R}{\partial q_n}, \dfrac{\partial R}{\partial p_1}, \dots, \dfrac{\partial R}{\partial p_n}\}$ must be of class C^1 and therefore the following relations must be true

$$\frac{\partial^2 R}{\partial q_i \partial q_j} = \frac{\partial^2 R}{\partial q_j \partial q_i}, \quad \frac{\partial^2 R}{\partial q_i \partial p_j} = \frac{\partial^2 R}{\partial p_j \partial q_i}, \quad \frac{\partial^2 R}{\partial p_i \partial p_j} = \frac{\partial^2 R}{\partial p_j \partial p_i} \tag{21}$$

To ease the notation, let the two sets of variables be denoted by

$$\begin{aligned} \{X_1, \ldots, X_{2n}\} &\equiv \{x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_n\} \\ \{Q_1, \ldots, Q_{2n}\} &\equiv \{q_1, \ldots, q_n, p_1, \ldots, p_n\} \quad , \end{aligned} \tag{22}$$

and let

$$X = \varphi(Q, t)$$

$$Q = \psi(X, t)$$

represent the given transformation and its inverse, respectively. Then, in summary: since $[\frac{\partial R}{\partial Q}]$ is a gradient of class C^1, Equations (21) must be satisfied. Thus, if there exists a solution $R(Q)$ of Equations (20), then $J[\frac{\partial \psi}{\partial t}]$ must be a gradient of class C^1, which implies that $\frac{\partial}{\partial Q}\{J[\frac{\partial \psi}{\partial t}]\}$ is a symmetric matrix. It will be shown that this is indeed the case.

By expanding each side of the following equality it is readily determined that

$$\frac{\partial}{\partial Q}\{J[\frac{\partial \psi}{\partial t}]\} = J\{\frac{\partial}{\partial Q}[\frac{\partial \psi}{\partial t}]\} \tag{23}$$

Another convenient representation is given by the following lemma.

Lemma 1: $\frac{\partial}{\partial Q}[\frac{\partial \psi}{\partial t}] = \frac{\partial N}{\partial t} N^{-1}$, where N is the Jacobian of the inverse transformation.

Proof: Consider the inverse transformation $Q = \psi(X, t)$. Then,

$$\frac{\partial\psi}{\partial X} = \frac{\partial\psi[\varphi(Q, t), t]}{\partial X} \quad \text{and} \quad \frac{\partial\psi}{\partial t} = \frac{\partial\psi[\varphi(Q, t), t]}{\partial t},$$

i.e., after the differentiations are performed the relation $X = \varphi(Q, t)$ is used to form a function of $\{Q, t\}$ again. Since $\psi(X, t) \in C^2$, it follows by the chain rule that

$$\frac{\partial}{\partial t}\left[\frac{\partial\psi}{\partial X}\right] = \frac{\partial}{\partial X}\left[\frac{\partial\psi}{\partial t}\right] = \frac{\partial}{\partial Q}\left[\frac{\partial\psi}{\partial t}\right]\frac{\partial\psi}{\partial X} \tag{24}$$

since $\frac{\partial\psi}{\partial t}$ is a function of $\{Q, t\}$. But, $N \equiv \frac{\partial\psi}{\partial X}$, so upon substitution in Equation (24), the following expression is obtained

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial Q}\left[\frac{\partial\psi}{\partial t}\right] N.$$

Then, since N^{-1} exists,

$$\frac{\partial}{\partial Q}\left[\frac{\partial\psi}{\partial t}\right] = \frac{\partial N}{\partial t} N^{-1}. \tag{25}$$

The representations of Equation (23) and Lemma 1 then give

$$\frac{\partial}{\partial Q}\left\{J\left[\frac{\partial\psi}{\partial t}\right]\right\} \equiv J \cdot \frac{\partial N}{\partial t} \cdot N^{-1}.$$

Thus, the problem is now to show that $J \cdot \frac{\partial N}{\partial t} \cdot N^{-1}$ is symmetric, i.e.,

$$\left[J \cdot \frac{\partial N}{\partial t} \cdot N^{-1}\right]^T = J \cdot \frac{\partial N}{\partial t} \cdot N^{-1}. \tag{26}$$

Since by hypothesis M is symplectic, it follows that N is symplectic since $N = M^{-1}$ and the symplectic matrices form a group.

Thus, $N^T JN = \frac{1}{\mu} J$. Since $\frac{1}{\mu} J$ is just a matrix of constants, it follows that

$$\frac{\partial}{\partial t}[\, N^T JN \,] = \frac{\partial}{\partial t}[\, \frac{1}{\mu} J \,] = 0$$

or,

$$(\frac{\partial N}{\partial t})^T J\dot{N} + N^T J \frac{\partial N}{\partial t} = 0.$$

Since J is skew-symmetric, $J^T = -J$ so

$$-(\frac{\partial N}{\partial t})^T J^T N + N^T J \frac{\partial N}{\partial t} = 0.$$

Upon multiplication by N^{-1} first on the right and then $(N^T)^{-1}$ on the left leads to the following expression

$$(N^{-1})^T (\frac{\partial N}{\partial t})^T J^T - J \frac{\partial N}{\partial t} N^{-1} = 0.$$

Therefore,

$$[\, J \frac{\partial N}{\partial t} N^{-1} \,]^T = J \frac{\partial N}{\partial t} N^{-1},$$

which verifies the symmetry of $J \frac{\partial N}{\partial t} N^{-1}$, and thus, there exists a solution $R(q, p, t)$ of Equations (20).

(Necessity →) Since $\{x(q, p, t), \lambda(q, p, t)\}$ is assumed to be canonical, there exists a $K(q, p, t)$ such that $\{\dot{q} = \frac{\partial K}{\partial p}, \dot{p} = -\frac{\partial K}{\partial q}\}$. It must be shown that M is symplectic (or, equivalently, that N is symplectic since $N = M^{-1}$ implies that M is symplectic if N is symplectic). Making use of the notation introduced in Equations (22), Equations (14) can be written as

$$\dot{Q} = M^{-1}\dot{X} + \frac{\partial Q}{\partial t} , \qquad (27)$$

where

$$Q = \begin{bmatrix} q \\ \\ p \end{bmatrix} \qquad X = \begin{bmatrix} x \\ \\ \lambda \end{bmatrix}$$

But, $\{H, x, \lambda\} \equiv \{H, X\}$ is a Hamiltonian system so

$$\dot{X} \equiv \begin{bmatrix} \frac{\partial H}{\partial \lambda} \\ \\ -\frac{\partial H}{\partial x} \end{bmatrix} = J \begin{bmatrix} \frac{\partial H}{\partial x} \\ \\ \frac{\partial H}{\partial \lambda} \end{bmatrix} = J[\frac{\partial H}{\partial X}].$$

Substitution in Equation (27) gives

$$\dot{Q} = M^{-1} J [\frac{\partial H}{\partial X}] + \frac{\partial Q}{\partial t} . \qquad (28)$$

But, $H(X, t) = H[X(Q, t), t]$ which implies (upon application of the chain rule) that

$$\frac{\partial H}{\partial Q} = [\frac{\partial X}{\partial Q}]^T \frac{\partial H}{\partial X} = M^T \frac{\partial H}{\partial X} .$$

Substitution in Equation (28) gives

$$\dot{Q} = M^{-1} J (M^T)^{-1} \frac{\partial H}{\partial Q} + \frac{\partial Q}{\partial t} .$$

It is given that $\{K, q, p\} \equiv \{K, Q\}$ is a Hamiltonian system, so $\dot{Q} = J \frac{\partial K}{\partial Q}$ which implies

$$J \frac{\partial K}{\partial Q} = M^{-1} J (M^T)^{-1} \frac{\partial H}{\partial Q} + \frac{\partial Q}{\partial t} . \tag{29}$$

Recalling that $J^{-1} = -J$ (Property C. 3),

$$\frac{\partial K}{\partial Q} = -JM^{-1} J (M^T)^{-1} \frac{\partial H}{\partial Q} - J \frac{\partial Q}{\partial t} ,$$

or

$$\frac{\partial K}{\partial Q} = -JNJN^T \frac{\partial H}{\partial Q} - J \frac{\partial Q}{\partial t} ,$$

where use has been made of the matrix identity $(M^T)^{-1} = (M^{-1})^T$.

Then,

$$\frac{\partial K}{\partial Q} = -J[NJN^T \frac{\partial H}{\partial Q} + \frac{\partial Q}{\partial t}]. \tag{30}$$

The left-hand side of Equation (30) is a gradient. It will now be shown that the right-hand side of Equation (30) is a gradient (for every Hamiltonian H) only if N is a symplectic matrix.

Since Equation (30) must hold for every H-function, in particular it must hold for $H \equiv 0$. Then,

$$\frac{\partial K}{\partial Q} = -J \frac{\partial Q}{\partial t} .$$

which implies that $-J\frac{\partial Q}{\partial t}$ is a gradient with respect to Q. It then follows that $JNJN^T(\frac{\partial H}{\partial Q})$ must be a gradient. On observing this fact, the following Lemma can be stated.

Lemma 2: If $JNJN^T\frac{\partial H}{\partial Q}$ is a gradient for every choice of H, then $JNJN^T = \mu I$, where μ = constant ($\neq 0$).

Proof: Let $A \equiv JNJN^T$. Then $A\frac{\partial H}{\partial Q}$ is a gradient. First, consider the 2n classes of Hamiltonian functions which are polynomials in only one $Q_i \in \{Q_1, \ldots, Q_{2n}\}$. Then, the following vectors are gradients:

$$\begin{bmatrix} a_{1,1} & g_1(Q_1) \\ a_{2,1} & g_1(Q_1) \\ & \vdots \\ a_{2n,1} & g_1(Q_1) \end{bmatrix}, \begin{bmatrix} a_{1,2} & g_2(Q_2) \\ a_{2,2} & g_2(Q_2) \\ & \vdots \\ a_{2n,2} & g_2(Q_2) \end{bmatrix}, \ldots, \begin{bmatrix} a_{1,2n} & g_{2n}(Q_{2n}) \\ a_{2,2n} & g_{2n}(Q_{2n}) \\ & \vdots \\ a_{2n,2n} & g_{2n}(Q_{2n}) \end{bmatrix} \quad (31)$$

where

$$A \equiv \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,2n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,2n} \\ \vdots & \vdots & & \vdots \\ a_{2n,1} & a_{2n,2} & \cdots & a_{2n,2n} \end{bmatrix}.$$

and

$$g_i(Q_i) \equiv \frac{\partial H(Q_i)}{\partial Q_i}. \qquad (i = 1, \ldots, 2n)$$

Since each of the vectors of Equations (31) is a gradient, there exist $2n$ functions $B_1(Q), \ldots, B_{2n}(Q)$ such that:

$$\begin{bmatrix} \frac{\partial B_i}{\partial Q_1} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial B_i}{\partial Q_{2n}} \end{bmatrix} = \begin{bmatrix} a_{1,i} & g_i(Q_i) \\ & \vdots \\ a_{2n,i} & g_i(Q_i) \end{bmatrix} . \quad (i = 1, \ldots, 2n) \tag{32}$$

Given a particular $i \in \{1, 2, \ldots, 2n\}$, Equation (32) can be viewed as an integrable system of first-order partial differential equations with dependent variable B_i. Thus, the integrability conditions must be satisfied (i.e., $\frac{\partial^2 B_i}{\partial Q_j \partial Q_k} = \frac{\partial^2 B_i}{\partial Q_k \partial Q_j}$). This is equivalent to the requirement that

$$\frac{\partial}{\partial Q} \begin{bmatrix} a_{1,i} & g_i(Q_i) \\ & \vdots \\ a_{2n,i} & g_i(Q_i) \end{bmatrix} \qquad (i = 1, \ldots, 2n)$$

be a symmetric matrix. Thus,

$$\frac{\partial(a_{i,k} g_k)}{\partial Q_j} = \frac{\partial(a_{j,k} g_k)}{\partial Q_i}$$

for each $i, j, k = 1, 2, \ldots, 2n$. Consider the case $i = k \neq j$:

$$\frac{\partial a_{i,k}}{\partial Q_j} g_k + a_{i,k} \frac{\partial g_k}{\partial Q_j} = \frac{\partial a_{j,k}}{\partial Q_i} g_k + a_{j,k} \frac{\partial g_k}{\partial Q_i} .$$

But, g_k depends on only Q_k and since $i = k$, $j \neq k$

$$\frac{\partial a_{k,k}}{\partial Q_j} g_k + 0 = \frac{\partial a_{j,k}}{\partial Q_k} g_k + a_{j,k} \frac{\partial g_k}{\partial Q_k} . \tag{33}$$

Suppose $H(Q_k)$ is a first-degree polynomial in Q_k. Then,

$$g_k \equiv \frac{\partial H}{\partial Q_k} = \text{constant} \rightarrow \frac{\partial g_k}{\partial Q_k} = 0.$$

Since Equation (33) must hold for all choices of H, it follows that

$$\frac{\partial a_{k,k}}{\partial Q_j} g_k = \frac{\partial a_{j,k}}{\partial Q_k} g_k$$

or,

$$\left(\frac{\partial a_{k,k}}{\partial Q_j} - \frac{\partial a_{j,k}}{\partial Q_k}\right) g_k = 0.$$

Since $g_k \neq 0$ in general, then

$$\frac{\partial a_{k,k}}{\partial Q_j} = \frac{\partial a_{j,k}}{\partial Q_k} . \qquad (j \neq k) \tag{34}$$

Substitution of Equation (34) into Equation (33) then shows that

$$a_{j,k} = 0 \qquad (j \neq k) \tag{35}$$

since $\frac{\partial g_k}{\partial Q_k} \neq 0$, in general.

From Equation (35) it follows that A must be a diagonal matrix, and then Equation (34) becomes

$$\frac{\partial a_{k,k}}{\partial Q_j} = 0\,, \qquad (j \neq k)$$

which implies either $a_{k,k} = a_{k,k}(Q_k)$ or $a_{k,k}$ = constant.

Finally, consider the class of Hamiltonian functions $H = Q_i Q_{i+1}$ $(i = 1, \ldots, 2n-1)$. Then the following vectors are gradients:

$$\begin{bmatrix} a_{1,1}(Q_1)\, Q_2 \\ a_{2,2}(Q_2)\, Q_1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ a_{2,2}(Q_2)\, Q_3 \\ a_{3,3}(Q_3)\, Q_2 \\ 0 \\ \cdot \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ a_{2n-1,2n-1}(Q_{2n-1})\, Q_{2n} \\ a_{2n,2n}(Q_{2n})\, Q_{2n-1} \end{bmatrix} \qquad (36)$$

Again Equations (33) must be satisfied, so operating on Equations (36)

$$\frac{\partial[\, a_{1,1}(Q_1)Q_2\,]}{\partial Q_2} = \frac{\partial[\, a_{2,2}(Q_2)Q_1\,]}{\partial Q_1}$$

$$\frac{\partial[\, a_{2,2}(Q_2)Q_3\,]}{\partial Q_3} = \frac{\partial[\, a_{3,3}(Q_3)Q_2\,]}{\partial Q_2}$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{\partial[\, a_{2n-1,2n-1}(Q_{2n-1})Q_{2n}\,]}{\partial Q_{2n}} = \frac{\partial[\, a_{2n,2n}(Q_{2n})Q_{2n-1}\,]}{\partial Q_{2n-1}}$$

These conditions imply that

$$a_{1,1}(Q_1) = a_{2,2}(Q_2)$$

$$a_{2,2}(Q_2) = a_{3,3}(Q_3)$$

$$\vdots \qquad\qquad \vdots$$

$$a_{2n-1,2n-1}(Q_{2n-1}) = a_{2n,2n}(Q_{2n}) .$$

Thus, $a_{1,1}(Q_1) = a_{2,2}(Q_2) = \ldots = a_{2n,2n}(Q_{2n})$. But the set $\{Q_1, Q_2, \ldots, Q_{2n}\}$ is independent, so each of the diagonal elements must be the same nonzero constant, i.e.,

$$a_{1,1} = a_{2,2} = \ldots = a_{2n,2n} \equiv \mu^* = \text{constant.}$$

Thus, from the above lemma

$$JNJN^T = \mu^* I ,$$

or

$$NJN^T = \mu^* J^{-1} = -\mu^* J \equiv \mu J$$

Therefore, N is symplectic and the theorem is proved.

The above theorem not only gives the important necessary and sufficient condition for a canonical transformation, but also a method for constructing the new Hamiltonian if one has a transformation defined by a symplectic Jacobian. Note that the definition of a new Hamiltonian is no problem if the transformation does not contain time explicitly since then $R \equiv 0$. To this end, the following proposition is considered.

Proposition: Let $\{H(x, \lambda, t), x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_n\}$ be a Hamiltonian system. Then,

$$\{H^*(x, \lambda), x_1, \ldots, x_{n+1}, \lambda_1, \ldots, \lambda_{n+1}\}$$

with $H^* \equiv H + \lambda_{n+1}$ and $x_{n+1} = t$ is an equivalent Hamiltonian system which does not contain time explicitly, but has $(n+1)$-degrees of freedom with $H^*(x, \lambda)$ as a constant of the motion.

Proof: For $i = 1, \ldots, n$, no change occurs, i.e.,

$$\dot{\lambda}_i = -\frac{\partial H^*}{\partial x_i} = -\frac{\partial H}{\partial x_i} \quad ; \quad \dot{x}_i = \frac{\partial H^*}{\partial \lambda_i} = \frac{\partial H}{\partial \lambda_i} .$$

Now, consider the $\dot{\lambda}_{n+1}$ equation:

$$\dot{\lambda}_{n+1} = -\frac{\partial H^*}{\partial x_{n+1}} = -\frac{\partial H}{\partial x_{n+1}} = -\frac{\partial H}{\partial t} .$$

But,

$$\frac{dH}{dt} = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial x_i} \dot{x}_i + \frac{\partial H}{\partial \lambda_i} \dot{\lambda}_i \right) + \frac{\partial H}{\partial t}$$

or,

$$\frac{dH}{dt} = \sum_{i=1}^{n} \left(-\dot{\lambda}_i \dot{x}_i + \dot{x}_i \dot{\lambda}_i \right) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} .$$

Thus,

$$\dot{\lambda}_{n+1} = -\frac{\partial H}{\partial t} = -\frac{dH}{dt}$$

or,

$$\frac{d}{dt}(\lambda_{n+1} + H) = 0.$$

Therefore, $H^* = H + \lambda_{n+1}$ is a constant of the motion.

Finally, consider the $\dot{x}_{n+1}$ equation:

$$\dot{x}_{n+1} = \frac{\partial H^*}{\partial \lambda_{n+1}} = 1 \quad \rightarrow \quad x_{n+1} = t + \text{constant}.$$

Let $x_{n+1} = 0$ when $t = 0$. Then, $x_{n+1} = t$, as desired.

Thus, the two systems represent the same physical problem except that with the new system only transformations between "conservative" systems need be considered.

Some expositions on canonical transformations do not make mention of the above necessary and sufficient condition, but instead say that a transformation is canonical if Hamiltonian form is preserved and that this is true if there exists a function F such that

$$\sum_{i=1}^{n} \lambda_i \dot{x}_i - H(x, \lambda, t) = \sum_{i=1}^{n} p_i \dot{q}_i - K(q, p, t) + \frac{dF}{dt}.$$

Presently, the motivation for this relation (which, physically, is a form of Hamilton's principle) will be shown. First, though, a simple example will show why the word "every" is emphasized in the definition of a canonical transformation adopted in this report (i.e., Definition I. 3. 2).

Example: Consider the following transformation

$$x_1 = p_2 \qquad \lambda_1 = q_1$$

$$x_2 = p_1 \qquad \lambda_2 = q_2$$

The Jacobian for this transformation is

$$M = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

and, hence,

$$M^TJM = \left[\begin{array}{cc|cc} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \neq \mu \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right].$$

Therefore, M is not symplectic. So, by definition, this transformation is not canonical. However, if the Hamiltonian is,

$$H(x, \lambda, t) = x_1 x_2$$

then,

$$K(q, p, t) = -p_1 p_2$$

is a new Hamiltonian which satisfies the requirements that the Hamiltonian form be preserved. That is:

$$\begin{array}{lll}
\dot{x}_1 = \dfrac{\partial H}{\partial \lambda_1} = 0 & \dot{p}_2 = 0 & \dot{p}_1 = \dfrac{-\partial K}{\partial q_1} = 0 \\[2ex]
\dot{x}_2 = \dfrac{\partial H}{\partial \lambda_2} = 0 & \dot{p}_1 = 0 & \dot{p}_2 = \dfrac{-\partial K}{\partial q_2} = 0 \\[2ex]
\dot{\lambda}_1 = \dfrac{-\partial H}{\partial x_1} = -x_2 \quad \rightarrow & \dot{q}_1 = -p_1 & \leftarrow \quad \dot{q}_1 = \dfrac{\partial K}{\partial p_1} = -p_2 \\[2ex]
\dot{\lambda}_2 = \dfrac{-\partial H}{\partial x_2} = -x_1 & \dot{q}_2 = -p_2 & \dot{q}_2 = \dfrac{\partial K}{\partial p_2} = -p_1 .
\end{array}$$

However, if $H = x_1^2$, then there does not exist a new Hamiltonian $K(q, p, t)$ which preserves Hamiltonian form under the given transformation.

I. 3B Generating Functions

Theorem I. 3. 1 yields a means for checking a given transformation to see if it is a canonical transformation. However, the theorem does not give a method for developing the transformation. The forthcoming discussion will present methods which can be used for such a purpose.

Consider a Lagrange problem in the calculus of variations where the integral to be minimized is

$$I = \int_{t_o}^{t_f} L(x, \dot{x}, t)dt .$$

The function $L(x, \dot{x}, t)$ is referred to as a Lagrangian function and the equations (Euler-Lagrange equations):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0 \qquad (i = 1, \ldots, n)$$

must be satisfied on an extremal. The following question is then raised: If $L'(X, \dot{X}, t)$ is the Lagrangian in the X-space where $X = X(x)$, what is the relation between $L(x, \dot{x}, t)$ and $L'(X, \dot{X}, t)$ if they are the Lagrangian functions for the same physical process? The next theorem answers this question.

Theorem I. 3. 2: Let $X = X(x)$ be a nonsingular transformation and let $L(x, \dot{x}, t)$ and $L'(X, \dot{X}, t)$ be Lagrangian functions in their respective coordinate systems. If L and L' differ at most by the total time derivative of some scalar function (say S), then L and L' will be Lagrangians for the same extremals.

Proof: By hypothesis, $L = L' + \frac{dS}{dt}$. Since both L and L' are Lagrangians, then

$$\delta \int_{t_o}^{t_f} L dt = 0, \qquad \delta \int_{t_o}^{t_f} L' dt = 0 .$$

Consider the problem defined by L. The extremals for this problem are determined by the solution of the set of Lagrange's equations for L, i.e.,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0 . \qquad (i = 1, \ldots, n) \tag{38}$$

If it can be shown that the extremals for the problem defined by L' are described by the differential equations (38), then the proof of the theorem is complete. Consider:

$$\delta \int_{t_o}^{t_f} L' dt = \delta \int_{t_o}^{t_f} \left(L - \frac{dS}{dt}\right) dt = \delta \int_{t_o}^{t_f} L dt - \delta \int_{S_o}^{S_f} dS = 0 .$$

But, $\delta \int_{S_o}^{S_f} dS = \delta [\, S_f - S_o \,] = \delta [\, \text{constant} \,] = 0$, so the extremals for the L'-problem are described by the differential equations which result from $\delta \int_{t_o}^{t_f} L dt = 0$ (i.e., Equations (38).).

In strict analogy with classical mechanics, the definition

$$H(x, \lambda, t) \equiv \sum_{i=1}^{n} \lambda_i \dot{x}_i - L(x, \dot{x}, t),$$

where $\lambda_i \equiv \partial L / \partial \dot{x}_i$, is the Hamiltonian associated with the Lagrangian L. Thus the following corollary to the above theorem can be obtained.

Corollary I. 3. 1: Let $L(x, \dot{x}, t)$ and $L'(q, \dot{q}, t)$ be Lagrangians for the same problem in two different coordinate systems connected by the nonsingular transformation $q = q(x)$. Define $\lambda_i \equiv \frac{\partial L}{\partial \dot{x}_i}$, $p_i \equiv \frac{\partial L'}{\partial \dot{q}_i}$ $(i = 1, \ldots, n)$ as the generalized momenta in the two coordinate systems. Then,

$$\sum_{i=1}^{n} \lambda_i \dot{x}_i - H(x, \lambda, t) = \sum_{i=1}^{n} p_i \dot{q}_i - K(q, p, t) + \frac{dS^*}{dt}, \tag{39}$$

where $K \equiv \sum_{i=1}^{n} p_i \dot{q}_i - L'$ and S^* is some scalar function.

Equation (39) represents a sufficient condition for a canonical transformation and is sometimes used as the definition for a canonical transformation since it is very useful in applications. The usefulness of Equation (39) is a consequence of the function S^*, which is called a generating function. That is, if S^* is given as a function of n of the $\{x, \lambda\}$-set and n of the $\{q, p\}$-set, none of which are conjugate pairs, then a canonical transformation is defined.

Example: Consider $S^* = \sum_{i=1}^{n} x_i q_i$ and let $\{x, q\}$ be the $2n$ independent variables of the $4n$ variables $\{x, \lambda, q, p\}$. Then, $\frac{dS^*}{dt} = \sum_{i=1}^{n} (\dot{x}_i q_i + x_i \dot{q}_i)$, so from Equation (39):

$$\sum_{i=1}^{n} \lambda_i \dot{x}_i - H = \sum_{i=1}^{n} p_i \dot{q}_i - K + \sum_{i=1}^{n} q_i \dot{x}_i + \sum_{i=1}^{n} x_i \dot{q}_i.$$

Thus, from the independence of the set $\{x, q, t\}$, it follows that

$$q_i = \lambda_i \qquad p_i = -x_i \qquad (i = 1, \ldots, n)$$

$$K(q, p) = H[\,x(q, p), \lambda(q, p)\,].$$

This canonical transformation is called the <u>reversal transformation</u> since it takes the old momenta into the new coordinates and the old coordinates into the negative of the new momenta.

In some expositions on the subject of canonical transformations, it is sometimes implied that there exist only four types of generating functions: $S^*(x, q, t)$, $S^*(x, p, t)$, $S^*(\lambda, q, t)$, $S^*(\lambda, p, t)$. Actually the class of generating functions is much larger and the following property is useful in applications.

<u>Proposition:</u> Let $z_i \epsilon \{x_i, \lambda_i\}$ and $Z_i \epsilon \{q_i, p_i\}$ for each $i = 1, 2, \ldots, n$. That is, neither two of the old nor two of the new variables can be conjugate to each other. The problem is to find the conditions which define a generating function $S(z_1, \ldots, z_n, Z_1, \ldots, Z_n, t)$. Note that the four cases mentioned above are special cases of this procedure. The conditions which define the transformation associated with $S(z, Z, t)$ are determined by Equation (39) and the following generating function:

$$S^* \equiv S(z, Z, t) + \sum_{i=1}^{n} x_i \lambda_i (z_i \Delta \lambda_i) - \sum_{i=1}^{n} q_i p_i (z_i \Delta p_i) \tag{40}$$

where it is convenient to define the operator $a \Delta b$ as follows:

$$a \Delta b \equiv \begin{cases} 1 & \text{if } a \equiv b \\ 0 & \text{if } a \neq b \end{cases} \tag{41}$$

Use of the operator is illustrated in the following example.

<u>Example:</u> Assume that

$\{x_1, \ldots, x_{n/2}, \lambda_{(n/2)+1}, \ldots, \lambda_n, p_1, \ldots, p_{n/2}, q_{(n/2)+1}, \ldots, q_n, t\}$

are to be treated as the $2n+1$ independent variables of the set $\{x, \lambda, q, p, t\}$. By the above proposition:

$$S^* = S(z, Z, t) + \sum_{i=\frac{n}{2}+1}^{n} x_i\lambda_i - \sum_{i=1}^{n/2} q_i p_i .$$

Then, by Equation (39):

$$\sum_{i=1}^{n} \lambda_i \dot{x}_i - H = \sum_{i=1}^{n} p_i \dot{q}_i - K + \sum_{i=1}^{n/2} \frac{\partial S}{\partial x_i} \dot{x}_i + \sum_{i=(n/2)+1}^{n} \frac{\partial S}{\partial \lambda_i} \dot{\lambda}_i$$

$$+ \sum_{i=1}^{n/2} \frac{\partial S}{\partial p_i} \dot{p}_i + \sum_{i=(n/2)+1}^{n} \frac{\partial S}{\partial q_i} \dot{q}_i + \frac{\partial S}{\partial t} + \sum_{i=(n/2)+1}^{n} (\dot{x}_i\lambda_i + x_i\dot{\lambda}_i)$$

$$- \sum_{i=1}^{n/2} (q_i\dot{p}_i + \dot{q}_i p_i).$$

After cancellation:

$$\sum_{i=1}^{n/2} \lambda_i \dot{x}_i - H = \sum_{i=\frac{n}{2}+1}^{n} p_i \dot{q}_i - K + \sum_{i=1}^{n/2} \frac{\partial S}{\partial x_i} \dot{x}_i + \sum_{i=\frac{n}{2}+1}^{n} \frac{\partial S}{\partial \lambda_i} \dot{\lambda}_i$$

$$+ \sum_{i=1}^{n/2} \frac{\partial S}{\partial p_i} \dot{p}_i + \sum_{i=\frac{n}{2}+1}^{n} \frac{\partial S}{\partial q_i} \dot{q}_i + \frac{\partial S}{\partial t} + \sum_{i=\frac{n}{2}+1}^{n} x_i \dot{\lambda}_i - \sum_{i=1}^{n/2} q_i \dot{p}_i .$$

This expression finally reduces to

$$\sum_{i=1}^{n/2} \left[\left(\lambda_i - \frac{\partial S}{\partial x_i}\right)\dot{x}_i + \left(q_i - \frac{\partial S}{\partial p_i}\right)\dot{p}_i \right] + \left(K - H - \frac{\partial S}{\partial t}\right)$$

$$+ \sum_{i=\frac{n}{2}+1}^{n} \left[-\left(p_i + \frac{\partial S}{\partial q_i}\right)\dot{q}_i - \left(x_i + \frac{\partial S}{\partial \lambda_i}\right)\dot{\lambda}_i \right] = 0.$$

Then, since $\{x_1, \ldots, x_{n/2}, p_1, \ldots, p_{n/2}, q_{n/2+1}, \ldots, q_n, \lambda_{n/2}, \ldots, \lambda_n, t\}$ is an independent set, the following conditions must hold:

$$\lambda_i = \frac{\partial S}{\partial x_i} \qquad q_i = \frac{\partial S}{\partial p_i} \qquad (i = 1, \ldots, \tfrac{n}{2})$$

$$p_i = -\frac{\partial S}{\partial q_i} \qquad x_i = -\frac{\partial S}{\partial \lambda_i} \qquad (i = \tfrac{n}{2}+1, \ldots, n) \tag{42}$$

$$K = H + \frac{\partial S}{\partial t}$$

Therefore, given any function which depends on the above mentioned independent variables, a canonical transformation is defined by Equations (42).

In applications use is frequently made of "simple" transformations, i.e., some of the momenta are switched to coordinates (and vice versa) while the remaining variables remain the same. Another simple property is applicable.

Proposition: Let $p_i \in \{x_i, \lambda_i\}$ for each $i = 1, \ldots, n$. That is, each of the new momenta will be either an old coordinate or an old momenta. Then,

$$S = \sum_{i=1}^{n} \{x_i p_i(\lambda_i \Delta p_i) - x_i q_i(x_i \Delta p_i)\} \tag{43}$$

defines the simple transformation.

Example: Consider the identity transformation, i.e., $p_i = \lambda_i$ for each i. Then,

$$S = \sum_{i=1}^{n} x_i p_i .$$

Since this generating function depends on $\{x, p\}$, the previous proposition is used to determine the conditions governing the transformation, i.e.,

$$S^* = S(x,p) + \sum_{i=1}^{n} x_i \lambda_i (x_i \Delta \lambda_i) - \sum_{i=1}^{n} q_i p_i (p_i \Delta p_i)$$

or,

$$S^* = S(x,p) - \sum_{i=1}^{n} q_i p_i \, .$$

Thus, Equation (39) becomes

$$\sum_{i=1}^{n} \lambda_i \dot{x}_i - H = \sum_{i=1}^{n} p_i \dot{q}_i - K + \sum_{i=1}^{n} \left(\frac{\partial S}{\partial x_i} \dot{x}_i + \frac{\partial S}{\partial p_i} \dot{p}_i - q_i \dot{p}_i - p_i \dot{q}_i \right).$$

On collecting the coefficients of like terms this reduces to

$$\sum_{i=1}^{n} \left\{ \left(\lambda_i - \frac{\partial S}{\partial x_i} \right) \dot{x}_i + \left(q_i - \frac{\partial S}{\partial p_i} \right) \dot{p}_i \right\} + (K - H) = 0.$$

Thus, the transformation is defined by the following relations.

$$\lambda_i = \frac{\partial S}{\partial x_i} = p_i ; \qquad q_i = \frac{\partial S}{\partial p_i} = x_i ; \qquad K = H ,$$

where

$$S = \sum_{i=1}^{n} x_i p_i \, .$$

I. 3C The Hamilton-Jacobi Equation

The previous section was concerned with the procedure for performing a canonical transformation when a generating function is given.

In this section attention will be given to the process of determining the generating function. Once the generating function is known, the canonical transformation can be performed immediately.

Let $\{H, x, \lambda\}$ be a given Hamiltonian system. If a canonical transformation to a new Hamiltonian system where $K \equiv 0$ can be effected, then the integration problem will be trivial, i.e.,

$$\begin{aligned} \dot{q}_i &= \frac{\partial K}{\partial p_i} = 0 & & q_i = \text{constant} \equiv \beta_i \\ & & \rightarrow & \\ \dot{p}_i &= -\frac{\partial K}{\partial q_i} = 0 & & p_i = \text{constant} \equiv \alpha_i . \end{aligned} \tag{44}$$

The Hamilton-Jacobi theory has as its fundamental objective, the definition of this particular canonical transformation.

Let $\{x, p, t\}$ be the subset of $2n+1$ independent variables of the set of $4n+1$ variables $\{x, \lambda, q, p, t\}$. From Equation (40):

$$S^* = S(x, p, t) - \sum_{i=1}^{n} q_i p_i .$$

Substitution of S^* in Equation (39) gives:

$$\sum_{i=1}^{n} \lambda_i \dot{x}_i - H = \sum_{i=1}^{n} p_i \dot{q}_i - K + \frac{dS}{dt} - \sum_{i=1}^{n} q_i \dot{p}_i - \sum_{i=1}^{n} p_i \dot{q}_i$$

or,

$$\sum_{i=1}^{n} \lambda_i \dot{x}_i - H = -K + \sum_{i=1}^{n} \left(\frac{\partial S}{\partial x_i} \dot{x}_i + \frac{\partial S}{\partial p_i} \dot{p}_i\right) + \frac{\partial S}{\partial t} - \sum_{i=1}^{n} q_i \dot{p}_i .$$

Then, since the set $\{x, p, t\}$ is independent

$$\lambda_i = \frac{\partial S}{\partial x_i}$$

$$q_i = \frac{\partial S}{\partial p_i} \qquad (i = 1, \ldots, n) \qquad (45)$$

$$K = H + \frac{\partial S}{\partial t} .$$

Thus, for the important special case when $K \equiv 0$, the third of Equations (45) yields the Hamilton-Jacobi equation (H-J equation):

$$H(x, \frac{\partial S}{\partial x}, t) + \frac{\partial S}{\partial t} = 0 , \qquad (46)$$

where the first set of Equations (45) has been used to replace λ_i by $\frac{\partial S}{\partial x_i}$ in the Hamiltonian. The H-J equation is a first-order partial differential equation which is to be solved for the generating function $S(x, \alpha, t)$, where $\alpha_i \equiv p_i$ in the $\{K \equiv 0, q_i \equiv \beta_i, p_i \equiv \alpha_i\}$ -system. As shown in the following important theorem, if a complete solution of the H-J equation can be determined, then a general solution to the original dynamical problem will be obtained.

Theorem I. 3. 3 (Jacobi's Theorem): Let $S(x, \alpha, t)$ be a complete solution of the H-J equation and $\{\beta\}$ be a set of n arbitrary constants, where $\beta_i = \frac{\partial S}{\partial \alpha_i}$. Then, the functions

$$x_i = x_i(\alpha, \beta, t) \qquad (i = 1, \ldots, n)$$

$$\lambda_i = \lambda_i(\alpha, \beta, t) = \frac{\partial S[x(\alpha, \beta, t), \alpha, t]}{\partial x_i} .$$

constitute a general solution of the original Hamilton's equations, i.e.,

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i}, \quad \dot{\lambda}_i = -\frac{\partial H}{\partial x_i}. \qquad (i = 1, \ldots, n)$$

Proof: Recall that the H-J equation can be used to define a generating function, S, for a canonical transformation from $\{H, x, \lambda\}$ to $\{K \equiv 0, q \equiv \beta, p \equiv \alpha\}$ where S is assumed to be dependent upon $\{x, p, t\}$. Thus, the transformation is governed by Equations (45), i.e.,

$$\lambda_i = \frac{\partial S}{\partial x_i}$$

$$\beta_i = \frac{\partial S}{\partial \alpha_i} \qquad (i = 1, \ldots, n)$$

$$0 = H + \frac{\partial S}{\partial t}.$$

Since S is a complete solution, it is a function of 2n independent parameters $\{\alpha_1, \ldots, \alpha_n\}$, so the system

$$\lambda_i = \frac{\partial S}{\partial x_i} = \lambda_i(x, \alpha, t)$$

$$\beta_i = \frac{\partial S}{\partial \alpha_i} = \beta_i(x, \alpha, t) \qquad (i = 1, \ldots, n)$$

represents 2n functions of the 2n+1 variables $\{x, \alpha, t\}$. Further, since S is a complete solution, it follows that $\left| \frac{\partial^2 S}{\partial x_i \partial \alpha_j} \right| \neq 0$. This allows the β_i-equations to be solved for the x_i's, i.e., $x_i = x_i(\alpha, \beta, t)$. Then,

$$x_i = x_i(\alpha, \beta, t)$$
$$\lambda_i = \lambda_i[\, x(\alpha, \beta, t), \alpha, t\,] \qquad (i = 1, \ldots, n)$$

represent a general solution of the original Hamilton's equations.

In summary then, three equivalent formulations for the optimal trajectory problem have been presented : (i) a set of n second-order ordinary differential equations (Lagrange's equations); (ii) a set of $2n$ first-order ordinary differential equations (Hamilton's equations); and (iii) a single first-order partial differential equation (H-J equation). In most instances, a system of ordinary differential equations is preferable to a partial differential equation. However, in an analytic analysis of the optimal trajectory problem this is not necessarily the case because of the elegant perturbation theories associated with the H-J equation. Before discussing Hamilton-Jacobi perturbation theory, another form of the H-J equation (useful in conservative systems) will be given.

Suppose that $\{H(x, \lambda), x, \lambda\}$ is the given Hamiltonian system, i.e., H does not contain t explicitly. Then, H is a constant of the motion:

$$\frac{dH}{dt} = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial x_i} \dot{x}_i + \frac{\partial H}{\partial \lambda_i} \dot{\lambda}_i \right) = \sum_{i=1}^{n} \left(-\dot{\lambda}_i \dot{x}_i + \dot{x}_i \dot{\lambda}_i \right) \equiv 0.$$

In this case it is sometimes advantageous to consider only generating functions which do not depend on time. Then from Equations (45)

$$\lambda_i = \frac{\partial S(x, p)}{\partial x_i}$$
$$q_i = \frac{\partial S(x, p)}{\partial p_i} \qquad (i = 1, \ldots, n) \tag{47}$$
$$K = H.$$

Instead of setting $K \equiv 0$, let the new Hamiltonian be any specified function of the new momenta, i.e., $K = K(p)$. Since K is a Hamiltonian

$$\dot{p}_i = \frac{-\partial K}{\partial q_i} = 0$$

$$\dot{q}_i = \frac{\partial K}{\partial p_i}$$

Thus p_i = constant = α_i ($i = 1, \ldots, n$) and so

$$\dot{q}_i = \frac{\partial K[p(\alpha)]}{\partial p_i} = \text{constant.}$$

Hence, once again the integration problem is trivial and the last of Equations (47) becomes

$$H(x, \frac{\partial S}{\partial x}) = K(\alpha) . \tag{48}$$

A special case of this equation is

$$H(x, \frac{\partial S}{\partial x}) = \alpha_1 , \tag{49}$$

which Born[6] calls the Hamilton-Jacobi equation.

I.4 Basic Hamiltonian Perturbation Theory

As it stands, the Hamilton-Jacobi theory is elegant but it does not solve many problems since it involves the integration of a partial differential equation. Thus, on the surface, it appears that little is gained by converting the original characteristic system (i.e., Hamilton's equations) into a partial differential equation (i.e., the H-J equation). However, in celestial mechanics approximate solutions to many nonlinear problems have been obtained by the application of perturbation theories based on the H-J equation. In the theory which follows, no small-parameter assumptions are made. If a small-parameter is present, use can be made of special techniques for such problems (e.g., von Zeipel's method[15], Poincare's small -parameter expansion method[16]) but they will not be presented here.

Instead of developing canonical perturbation theory and Hamilton-Jacobi perturbation theory seperately, they will be derived together since the derivations are essentially the same. Moreover, when these techniques are applied, it may be advantageous to use a combination of the two. The basic idea in both procedures is to make the integration problem trivial by performing a sequence of transformations which converge to "natural" variables for the problem (e.g., a set of canonic constants).

Let $\{H(x, \lambda, t), x, \lambda\}$ be a Hamiltonian system. Suppose that

$$H = H_o - \sum_{i=1}^{n} H_i ,$$

where a complete solution of the H-J equation for H_o is known. In practice, the finite sum is sometimes replaced by an infinite sum (e.g., a power series or Fourier expansion for $H-H_o$) but the procedure is the same as for a finite sum.

Since the H-J theory assumes the set $\{x, p, t\}$ is the independent set, the general equations for a canonical transformation are Equations (45), i.e.,

$$\lambda_i = \frac{\partial S}{\partial x_i}$$

$$q_i = \frac{\partial S}{\partial p_i} \qquad (i = 1, \ldots, n) \tag{50}$$

$$K = \frac{\partial S}{\partial t} + H = \left(\frac{\partial S}{\partial t} + H_o\right) - \sum_{i=1}^{n} H_i .$$

Let $S^o\ (x, \alpha, t)$ be a complete solution of the H-J equation for H_o:

$$\frac{\partial S^o}{\partial t} + H_o\left(x, \frac{\partial S^o}{\partial x}, t\right) = 0 , \tag{51}$$

and let the system

$$x_i^o = x_i^o(\alpha, \beta, t)$$
$$\qquad (i = 1, \ldots, n)$$
$$\lambda_i^o = \lambda_i^o(\alpha, \beta, t)$$

be the general solution of Hamilton's equations for H_o, where the set $\{\alpha, \beta\}$ is the set of canonic constants determined by the solution of Equation (51). From the last of Equations (50)

$$K = 0 - \sum_{i=1}^{n} H_i ,$$

or

$$K(\alpha, \beta, t) \equiv -\sum_{i=1}^{n} H_i[\, x^o(\alpha, \beta, t), \lambda^o(\alpha, \beta, t), t\,] \tag{52}$$

and

$$\dot{\beta}_i = \frac{\partial K}{\partial \alpha_i} \qquad \dot{\alpha}_i = -\frac{\partial K}{\partial \beta_i} . \qquad (i = 1, \ldots, n) \tag{53}$$

Thus, the result is a new Hamiltonian system $\{K, \alpha, \beta\}$.

There are two basic ways of attacking the "new" Hamiltonian problem defined by Equations (52) and (53). Canonical perturbation theory involves the integration of Equations (53) whereas H-J perturbation theory involves the integration of the H-J equation for the Hamiltonian K.

Define $K_o(\alpha, \beta, t) \equiv -H_1[x^o(\alpha, \beta, t), \lambda^o(\alpha, \beta, t), t]$. Then,

$$K = K_o - \sum_{i=2}^{n} H_i .$$

Consider the H-J equation for K_o:

$$\frac{\partial S^1}{\partial t} + K_o(\beta, \frac{\partial S^1}{\partial \beta}, t) = 0. \tag{54}$$

Let $S^1(a, \beta, t)$ be a complete solution of Equation (54). Applying the general canonical transformation Equations (50) again leads to the following expressions.

$$\alpha_i = \frac{\partial S^1}{\partial \beta_i}$$

$$b_i = \frac{\partial S^1}{\partial a_i} \qquad (i = 1, \ldots, n)$$

$$K^* = \frac{\partial S^1}{\partial t} + K = (\frac{\partial S^1}{\partial t} + K_o) - \sum_{i=2}^{n} H_i = -\sum_{i=2}^{n} H_i .$$

The set $\{a, b\}$ is a set of canonic constants for the problem defined by $H_o - H_1$, and from Jacobi's theorem, the set of equations

$$\beta_i = \beta_i(a, b, t) \qquad (i = 1, \ldots, n)$$

$$\alpha_i = \alpha_i(a, b, t)$$

represents a general solution of the Hamilton's equations (53). If $H_o - H_1$ is a valid approximation to the total Hamiltonian H, then the system

$$x_i^1(a, b, t) \equiv x^o[\alpha(a, b, t), \beta(a, b, t), t]$$
$$\lambda_i^1(a, b, t) \equiv \lambda^o[\alpha(a, b, t), \beta(a, b, t), t] \qquad (i = 1, \ldots, n)$$

should be a valid approximation to the general solution of the Hamilton's equations for the total Hamiltonian.

If the effects of $\sum_{i=2}^{n} H_i$ are required, the same procedure can be applied to H_2, H_3, etc. One of the most powerful aspects of a Hamiltonian perturbation theory is that one need not start all over when a higher order approximation or the effect of a new perturbation is required.

PART II.

APPLICATION OF CANONICAL TRANSFORMATION THEORY TO THE OPTIMAL LOW-THRUST TRANSFER

In the following discussion, the theory outlined in Part I will be applied to the problem of obtaining approximate analytical solutions to the optimal low-thrust trajectory problem. A base solution which represents the total solution of the coast-arc problem (i.e., the optimal trajectory problem when thrust is zero) is obtained for both the two dimensional polar representation of the optimal trajectory and for a three dimensional spherical representation. The time rates of change of the base canonic constants for the planar problem when the thrust effects are included are determined also.

II. 1 Introduction

Before the theory of Part I can be applied, the optimal trajectory problem must be expressed as a well-defined Hamiltonian system. That is, the given variational problem must be reduced to a system of first-order ordinary differential equations defined by a Hamiltonian function and a set of $2n+2$ boundary conditions.

Consider the problem of extremizing the integral

$$I = \int_{t_o}^{t_f} G(x, t)dt, \tag{1}$$

subject to the constraints

$$\dot{x}_i - f_i(x, u, t) = 0, \qquad (i = 1, \ldots, n) \tag{2}$$

and the geometric boundary conditions

$$x_i(t_o) = X_{io} \qquad (i = 1, \ldots, n) \tag{3}$$

$$M_i(x_f, t_f) = 0, \qquad (i = 1, \ldots, p \leq n) \tag{4}$$

where x is a n-vector of state variables and u is a m-vector of control variables. The problem can be formulated as a Lagrange problem[10] in the calculus of variations by introducing a set of unknown multipliers $\lambda_1, \ldots, \lambda_n$, and forming the augmented functional

$$I = \int_{t_o}^{t_f} [\, G(x, t) + \sum_{i=1}^{n} \lambda_i(\dot{x}_i - f_i)]dt, \tag{5}$$

If I is to be an extremal with respect to the choice of $u(t)$, the following necessary conditions must be satisfied:

(i) Lagrange's equations must be satisfied everywhere in the interval, $t_o \leq t \leq t_f$, i.e.,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0 \qquad (i = 1, \ldots, n) \tag{6}$$

$$\frac{\partial L}{\partial u_i} = 0, \qquad (i = 1, \ldots, m) \tag{7}$$

where

$$L \equiv G(x, t) + \sum_{i=1}^{n} \lambda_i(\dot{x}_i - f_i);$$

and

(ii) a set of transversality conditions, say:

$$N_i(x_f, \lambda_f, t_f) = 0 \qquad (i = 1, \ldots, n-p+1) \qquad (8)$$

must be satisfied at the terminal time.

The system of Equations (2), (6), and (7) can be expressed as a system of first-order equations by defining a generalized Hamiltonian function

$$H^*(x, u, \lambda, t) \equiv \sum_{i=1}^{n} \lambda_i \dot{x}_i - L(x, \dot{x}, u, \lambda, t), \qquad (9)$$

and then developing Hamilton's equations, i.e.,

$$\dot{x}_i = \frac{\partial H^*}{\partial \lambda_i} \qquad \dot{\lambda}_i = -\frac{\partial H^*}{\partial x_i}. \qquad (i = 1, \ldots, n) \qquad (10)$$

Equation (7) and Equation (9) can be combined to yield.

$$\frac{\partial H^*}{\partial u_i} = 0. \qquad (i = 1, \ldots, m) \qquad (11)$$

In addition, the Weierstrass condition[10] must be satisfied if the functional defined by Equation (1) is to be a minimum. This leads to the further requirement that

$$\sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\partial^2 H^*}{\partial u_i \partial u_j} \delta u_i \delta u_j \geq 0, \qquad (12)$$

for all admissible δu_i and δu_j. If the inequality holds in Equation (12), the extremal is a minimizing trajectory. If the equality holds over any portion of the trajectory, that portion is referred to as a singular arc. In the subsequent discussion it is assumed that Equation (12) is a strict inequality for all $t_o \le t \le t_f$.

In most optimal trajectory problems, Equations (11) and (12) can be used to express the control variables as functions of the state variables and the Lagrange multipliers, say:

$$u_i = g_i(x, \lambda) . \qquad (i = 1, \ldots, m) \tag{13}$$

Consider the composite function

$$H(x, \lambda, t) \equiv H^*[x, \lambda, g(x, \lambda), t] . \tag{14}$$

By Equations (11), it follows that:

$$\frac{\partial H}{\partial x_i} = \frac{\partial H^*}{\partial x_i} + \sum_{j=1}^{m} \frac{\partial H^*}{\partial u_j} \frac{\partial g_j}{\partial x_i} = \frac{\partial H^*}{\partial x_i} = -\dot{\lambda}_i \tag{15}$$

$$\frac{\partial H}{\partial \lambda_i} = \frac{\partial H^*}{\partial \lambda_i} + \sum_{j=1}^{m} \frac{\partial H^*}{\partial u_j} \frac{\partial g_j}{\partial \lambda_i} = \frac{\partial H^*}{\partial \lambda_i} = \dot{x}_i \tag{16}$$

since $\partial H^*/\partial u_j = 0$ for $j = 1, \ldots, m$. Thus, Equations (15) and (16) are Hamilton's equations for the Hamiltonian of Equation (14). Equation (14) is a function of only the n state variables, the n Lagrange multipliers, and time.

Therefore, Equations (15) and (16) along with the boundary conditions of Equations (3), (4), and (8) represent a well-defined Hamiltonian system described by the "generalized coordinates" $\{x_1, \ldots, x_n\}$ and the "generalized momenta" $\{\lambda_1, \ldots, \lambda_n\}$. Thus, the classical perturbation theories of Hamiltonian mechanics are now available for the optimal trajectory problem.

II. 2 The Planar Problem

Consider the problem of minimizing the time of flight of a vehicle powered by a continuously-thrusting engine where the thrust and mass-flow rate are assumed constant. Although not a necessary assumption in what follows, it will generally be assumed that the thrust is small when compared with the gravitational force of attraction (e.g., a near-earth low-thrust mission). In this section, the state of the vehicle will be described by a polar coordinate system.

II. 2A The Planar Base Solution

The equations of motion for the vehicle are (see Fig. 1):

$$\dot{r} = u \qquad \dot{u} = \frac{v^2}{r} - \frac{\mu}{r^2} + \frac{F}{m}\sin\alpha$$
$$\dot{\theta} = \frac{v}{r} \qquad \dot{v} = -\frac{uv}{r} + \frac{F}{m}\cos\alpha \tag{17}$$

$$m = m_o + \dot{m}_o(t - t_o). \tag{18}$$

The generalized Hamiltonian function can be written as

$$H^* \equiv \sum_{i=1}^{4} \lambda_i f_i(x, \alpha, t), \tag{19}$$

where the functions $f_i(x, \alpha, t)$ $(i = 1, \ldots, 4)$ represent the right-hand sides of Equations (17) and the variables x_i $(i = 1, \ldots, 4)$ represent the state variables r, θ, u, v. The control, α, can be expressed as a function of the

Lagrange multipliers by applying Equations (11) and (12) to the Hamiltonian of Equation (19), i.e.,

$$\cos\alpha = \frac{-\lambda_4}{\sqrt{\lambda_3^2 + \lambda_4^2}} \qquad \sin\alpha = \frac{-\lambda_3}{\sqrt{\lambda_3^2 + \lambda_4^2}} . \tag{20}$$

Then, on substitution of Equations (20) into Equation (19), a new Hamiltonian $H(x, \lambda, t)$ is determined

$$H(x, \lambda, t) = H^*[x, \lambda, \alpha(\lambda), t] \tag{21}$$

or

$$H(x, \lambda, t) = [\lambda_1 u + \lambda_2 \frac{v}{r} + \lambda_3 (\frac{v^2}{r} - \frac{\mu}{r^2}) - \lambda_4 \frac{uv}{r}] - \frac{F}{m}\sqrt{\lambda_3^2 + \lambda_4^2} . \tag{22}$$

Equation (22) can be partitioned into a base Hamiltonian H_o and a perturbing Hamiltonian, $\frac{F}{m}\sqrt{\lambda_3^2 + \lambda_4^2}$, i.e.,

$$H \equiv H_o - \frac{F}{m}\sqrt{\lambda_3^2 + \lambda_4^2} . \tag{23}$$

If the thrust, F, is zero, then $H = H_o$. Physically, H_o is the variational Hamiltonian for the coast-arc problem and it is well known that closed-form solutions exist for this problem [17, 18, 19]. However, to apply canonical perturbation theory one must have a solution to the Hamilton-Jacobi (H-J) equation for this problem in terms of eight canonic constants.

Before writing the H-J equation for H_o, it will be advantageous to consider a physical interpretation of the problem. Since thrust is zero, the state variables represent a Keplerian orbit and thus, are not affected by the Lagrange multipliers. Hence, for the base problem, the energy and angular momentum of the orbit should be constants of the motion. Also, H_o does not contain time explicitly, so H_o should be a constant of the motion; and θ does not appear explicitly in H_o, so the conjugate variable for θ should be a constant of the motion. Thus, four constants of the motion are readily apparent. To apply the H-J theory most effectively, a simple canonical transformation should be used to make the above-mentioned constants of the motion the new momenta, i.e., $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, since then the four remaining constants of the motion will follow by differentiation of the generating function.

If the H-J equation for H_o is written with the x_i as generalized coordinates and the λ_i as generalized momenta (as is the case in Equation (22)), then advantage cannot be taken of the constantsof the motion mentioned above. However, if half of the momenta are switched to coordinates and vice versa by use of a simple transformation (see Equation (I. 43)), i.e.,

$$S = rp_1 + \theta p_2 - uq_1 - vq_2$$

with

$$\begin{aligned} p_1 &= \lambda_1 & q_1 &= r \\ p_2 &= \lambda_2 & q_2 &= \theta \\ p_3 &= u & q_3 &= -\lambda_3 \\ p_4 &= v & q_4 &= -\lambda_4 \end{aligned} \qquad (24)$$

then full advantage can be taken of the known constants of the motion. Substitution of Equations (24) into H_o results in the following Hamiltonian

$$H_o'(q, p) = p_1 p_3 + \frac{p_2 p_4}{q_1} - q_3\left(\frac{p_4^2}{q_1} - \frac{\mu}{q_1^2}\right) + \frac{q_4 p_3 p_4}{q_1} . \tag{25}$$

In order to find a base solution, a canonical transformation from the {q, p} set to a new {Q, P} set must be performed such that the new variables will be constants of the motion. The generating function for such a transformation can be obtained by solving the H-J equation.

The H-J equation for H_o' is

$$\frac{\partial S}{\partial t} + H_o'\left(q, \frac{\partial S}{\partial q}\right) = 0 . \tag{26}$$

Since t and q_2 only appear once each in Equation (26), and only in the form $\frac{\partial S}{\partial t}$ and $\frac{\partial S}{\partial q_2}$, it is reasonable to assume a partial separation of variables for S, i.e.,

$$S = S_1(t) + S_2(q_2) + S^*(q_1, q_3, q_4) . \tag{27}$$

By substituting Equation (27) into Equation (26) and using independence arguments, the following expressions are obtained

$$\frac{\partial S}{\partial t} = \frac{\partial S_1}{\partial t} = \alpha_1 , \qquad \frac{\partial S}{\partial q_2} = \frac{\partial S_2}{\partial q_2} = \alpha_2 , \tag{28}$$

where α_1 and α_2 are constants. Substitution of Equations (28) into Equation (27) leads to

$$S = \alpha_1 t + \alpha_2 q_2 + S^*(q_1, q_3, q_4) . \tag{29}$$

For a complete solution of Equation (26), four independent constants are required. Thus, two more constants are necessary. As previously mentioned, knowledge of the two-body problem could be used to define the other two, i.e., the energy and angular momentum of the orbit. However, these constants can also be obtained by inspecting the characteristic system of the H-J equation, which is now a function of only three variables, i.e.,

$$\frac{dq_i}{d\tau} = \frac{\partial F}{\partial p_i} ; \qquad \frac{dp_i}{d\tau} = -\frac{\partial F}{\partial q_i} , \qquad (i = 1,3,4) \tag{30}$$

where τ is an arbitrary parameter. Note that for the case $\tau = t$, the characteristic equations are just the Hamilton's equations associated with the H-J equation.

To obtain the angular momentum integral, consider the $\frac{dq_1}{d\tau}$ and $\frac{dp_4}{d\tau}$ equations for the characteristic system, i.e.,

$$\frac{dq_1}{d\tau} = p_3 ; \qquad \frac{dp_4}{d\tau} = -\frac{p_3 p_4}{q_1} . \tag{31}$$

Note that

$$\frac{dp_4}{d\tau} = \frac{dp_4}{dq_1}\frac{dq_1}{d\tau} = \frac{dp_4}{dq_1} p_3 . \tag{32}$$

so from Equations (31): $\frac{dp_4}{dq_1} = -\frac{p_4}{q_1}$. Substitution of this result into Equation (32) leads to a simple integration which gives

$$\alpha_3 = q_1 p_4 \, . \qquad (32)$$

To obtain the energy integral, Equation (33) is used along with the characteristic equations

$$\frac{dq_1}{d\tau} = p_3 ; \qquad \frac{dp_3}{d\tau} = \frac{\alpha_3^2}{q_1^3} - \frac{\mu}{q_1^2} \, . \qquad (34)$$

Noting that

$$\frac{dp_3}{d\tau} = \frac{dp_3}{dq_1} \frac{dq_1}{d\tau} = \frac{dp_3}{dq_1} \; p_3$$

and obtaining $\frac{dp_3}{dq_1}$ by eliminating τ in Equations (34) leads to another simple integration which gives

$$\alpha_4 = -p_3^2 - \frac{\alpha_3^2}{q_1^2} + \frac{2\mu}{q_1} \, . \qquad (35)$$

The four constants required for the complete solution have now been obtained. The only remaining problem is to incorporate them into the generating function of Equation (27). This can be effected by considering the following integrable Pfaff differential equation

$$dS^* = \frac{\partial S^*}{\partial q_1} dq_1 + \frac{\partial S^*}{\partial q_3} dq_3 + \frac{\partial S^*}{\partial q_4} dq_4 \, . \qquad (36)$$

where

$$\begin{aligned} \frac{\partial S^*}{\partial q_3} &= p_3(q_1) = \pm \frac{1}{q_1} \sqrt{-\alpha_3^2 + 2\mu q_1 - \alpha_4 q_1^2} \\ \frac{\partial S^*}{\partial q_4} &= p_4(q_1) = \frac{\alpha_3}{q_1} \end{aligned} \qquad (37)$$

and where $\frac{\partial S^*}{\partial q_1}(q_1, q_3, q_4)$ is defined by substituting Equations (37) into Equation (26); that is, $\frac{\partial S^*}{\partial q_1}$ is defined by satisfaction of the H-J equation. As shown in Section I. 2B, since $\frac{\partial S^*}{\partial q_3}$ and $\frac{\partial S^*}{\partial q_4}$ depend only on q_1, the generating function S^* is necessarily of the form

$$S^* = p_3(q_1)q_3 + p_4(q_1)q_4 + S'(q_1) , \tag{38}$$

where $S'(q_1)$ is to be determined. In the first of Equations (37), a $\pm$ sign is included since Equation (35) is an equation involving p_3^2. Physically, this corresponds to the radius increasing from perigee to apogee ($p_3 \equiv \dot{r} \geq 0$) and decreasing from apogee to perigee ($p_3 \equiv \dot{r} \leq 0$).

To determine $S'(q_1)$, consider $\frac{\partial S^*}{\partial q_1}$ as defined by the H-J equation. From the argument leading to the form of Equation (38), the terms of $\frac{\partial S^*}{\partial q_1}$ which contain q_3 and q_4 can be omitted, so then

$$S'(q_1) = \int \left(\frac{\partial S^*}{\partial q_1}\right)' dq_1 \quad , \tag{39}$$

where $\left(\frac{\partial S^*}{\partial q_1}\right)'$ is that portion of $\frac{\partial S^*}{\partial q_1}$ which does not contain q_3 or q_4. Integration of Equation (39) yields

$$S'(q_1) = \frac{\alpha_1 q_1 p_3(q_1)}{\alpha_4} \pm \frac{\mu\alpha_1}{\alpha_4^{3/2}} \sin^{-1}\left[\frac{\mu - \alpha_4 q_1}{\sqrt{\mu^2 - \alpha_4\alpha_3^2}}\right] \pm \alpha_2 \sin^{-1}\left[\frac{\mu q_1 - \alpha_3^2}{q_1\sqrt{\mu^2 - \alpha_4\alpha_3^2}}\right] \tag{40}$$

The integration of Equation (39) involves an assumption on the sign of α_4

(the negative of the energy) and $\alpha_4 > 0$ was assumed for the result given in Equation (40). This assumption restricts the solution to trajectories where the energy is negative (i.e., circular and elliptical conditions). Other $S'(q_1)$ functions can also be integrated for the parabolic and hyperbolic cases.

Consideration of Equations (29), (38), and (40) then gives the generating function for the base solution:

$$
\begin{aligned}
S = \alpha_1 t + \alpha_2 q_2 + p_3(q_1, \alpha_3, \alpha_4) q_3 + p_4(q_1, \alpha_3, \alpha_4) q_4 \\
+ S'(q_1, \alpha_1, \alpha_2, \alpha_3, \alpha_4) .
\end{aligned}
\tag{41}
$$

By Jacobi's Theorem, the remaining canonic constants of the motion are obtained by differentiating the generating function with respect to each of the α_i's, i.e., $\beta_i \equiv \frac{\partial S}{\partial \alpha_i}$. As functions of the original variables (see Equations (24)) the total set of canonic constants are:

$$
\begin{aligned}
\alpha_1 &= -\lambda_1 u - \lambda_2 \frac{v}{r} - \lambda_3 \left(\frac{v^2}{r} - \frac{\mu}{r^2}\right) + \lambda_4 \frac{uv}{r} \\
\alpha_2 &= \lambda_2 \\
\alpha_3 &= rv \\
\alpha_4 &= -u^2 - v^2 + \frac{2\mu}{r} \\
\beta_1 &= \frac{\partial S}{\partial \alpha_1} = t + \frac{ru}{\alpha_4} + \frac{u}{|u|} \frac{\mu}{\alpha_4^{3/2}} \sin^{-1}\left[\frac{\mu - \alpha_4 r}{\sqrt{\mu^2 - \alpha_4 \alpha_3^2}}\right] \\
\beta_2 &= \frac{\partial S}{\partial \alpha_2} = \theta - \frac{u}{|u|} \sin^{-1}\left[\frac{\mu r - \alpha_3^2}{r\sqrt{\mu^2 - \alpha_4 \alpha_3^2}}\right] \\
\beta_3 &= \frac{\partial S}{\partial \alpha_3} = -\frac{\lambda_4}{r} + \frac{\alpha_3 \lambda_3}{r^2 u} + \frac{\lambda_2}{ru} + \frac{\alpha_1 \alpha_3 (\alpha_3^2 - \mu r) + \mu \lambda_2 (\mu - \alpha_4 r)}{ru(\mu^2 - \alpha_4 \alpha_3^2)}
\end{aligned}
\tag{42}
$$

$$\beta_4 = \frac{\partial S}{\partial \alpha_4} = \frac{\lambda_3}{2u} - \frac{\alpha_1 r}{\alpha_4}\left(\frac{u}{\alpha_4} + \frac{1}{2u}\right) - \frac{u}{|u|}\,\frac{3}{2}\,\frac{\alpha_1 \mu}{\alpha_4^{5/2}} \sin^{-1}\left[\frac{\mu - \alpha_4 r}{\sqrt{\mu^2 - \alpha_4 \alpha_3^2}}\right]$$

$$+ \frac{\mu \alpha_1 (\alpha_3^2 \alpha_4 r - 2\mu^2 r + \mu \alpha_3^2) + \alpha_2 \alpha_3 \alpha_4^2 (\alpha_3^2 - \mu r)}{2\alpha_4^2 r u (\mu^2 - \alpha_4 \alpha_3^2)} \qquad (42)$$

The set $\{\alpha, \beta\}$ represents the closed-form solution to the coast-arc problem. The subset $\{\alpha_3, \alpha_4, \beta_1, \beta_2\}$ defines the Keplerian orbit, and the subset $\{\alpha_1, \alpha_2, \beta_3, \beta_4\}$ defines the solution for the Lagrange multipliers on a coast-arc. However, Equations (42) possess two singularities: $\dot{r} \equiv u = 0$ and energy $= -\alpha_4 = 0$. Thus, the class of missions to which the set $\{\alpha, \beta\}$ is applicable is somewhat restricted (i.e., elliptical trajectories with nonzero eccentricity). Recent attempts to obtain another base solution free of the $\dot{r} = 0$ singularity have been successful and the new solution is now being studied. The results of this analysis will be presented at a later date.

II. 2B The Canonic Perturbation Equations

The solution to the problem defined by the total Hamiltonian of Equation (23) may now be attacked by canonical perturbation theory. The perturbing Hamiltonian is

$$-H_1 = -\frac{F}{m}\sqrt{\lambda_3^2 + \lambda_4^2} \qquad (43)$$

To develop the canonic perturbation equations, i.e., the time rates of change of the base canonic constants, the perturbing Hamiltonian must be expressed as a function of the canonic constants and time. Thus, the canonic constant expressions $\{\alpha(x, \lambda, t), \quad \beta(x, \lambda, t)\}$ of Equations (42) must be inverted to give $\{x(\alpha, \beta, t), \lambda(\alpha, \beta, t)\}$. Such an inversion is not possible in closed form since the β_1-equation is a form of Kepler's equation. However, the original variables $\{x, \lambda\}$ may be expressed as functions of the canonic constants and the radius, so the implicit function theorem may be employed to give the desired perturbation equations.

Since the perturbing Hamiltonian is only a function of λ_3, λ_4, and m, only the β_1, β_3, and β_4 equations need to be inverted to obtain $\{\lambda_3(\alpha, \beta, r), \lambda_4(\alpha, \beta, r), t(r, \alpha, \beta)\}$. The β_1-equation actually defines the implicit relationship $r(t)$ which will be studied later.

Recalling from the first of Equations (37) that $p_3(\alpha_3, \alpha_4, r)$, the β_3-equation can be rearranged to give

$$-\lambda_4 \equiv q_4 = (q_1)\beta_3 + \left(\frac{\alpha_3}{q_1 p_3}\right) q_3 - \left[\frac{1}{p_3} + \frac{\mu^2 - \mu\alpha_4 q_1}{p_3(\mu^2 - \alpha_4\alpha_3^2)}\right]\alpha_2$$

$$-\left[\frac{\alpha_3(\alpha_3^2 - \mu q_1)}{p_3(\mu^2 - \alpha_4\alpha_3^2)}\right]\alpha_1$$

or,

$$q_4 \equiv A_1(q_1)\beta_3 + A_2(q_1, \alpha_3, \alpha_4)q_3 + A_3(q_1, \alpha_3, \alpha_4)\alpha_2 + A_4(q_1, \alpha_3, \alpha_4)\alpha_1. \tag{44}$$

Rearrangement of the β_4-equation gives

$$-\lambda_3 \equiv q_3 = (-2p_3)\beta_4 + \left[\frac{\alpha_3(\alpha_3^2 - \mu q_1)}{q_1(\mu^2 - \alpha_4\alpha_3^2)}\right]\alpha_2 + \left\{\left[\frac{-3\mu\sqrt{-\alpha_3^2 + 2\mu q_1 - \alpha_4 q_1^2}}{q_1(\alpha_4)^{5/2}}\right]\cdot\right.$$

$$\left.\cdot \sin^{-1}\left(\frac{\mu - \alpha_4 q_1}{\sqrt{\mu^2 - \alpha_4\alpha_3^2}}\right) + \frac{(\mu^2\alpha_4 - \alpha_3^2\alpha_4^2)q_1^2 + (5\mu\alpha_3^2\alpha_4 - 6\mu^3)q_1 + \alpha_3^2(3\mu^2 \quad 2\alpha_3^2\alpha_4}{\alpha_4^2 q_1(\mu^2 - \alpha_4\alpha_3^2)}\right\}$$

or,

$$q_3 \equiv B_1(q_1, \alpha_3, \alpha_4)\beta_4 + B_2(q_1, \alpha_3, \alpha_4)\alpha_2 + B_3(q_1, \alpha_3, \alpha_4)\alpha_1 . \qquad (45)$$

In many optimal trajectory problems (assuming a fixed initial state) the final value of θ is arbitrary. Thus, from the transversality conditions, $\lambda_2 = \alpha_2 = 0$ for such a mission. Then, Equation (44) would lose the A_3-term and Equation (45) would lose the B_2-term.

Equation (45) is of the form $q_3(\alpha, \beta, q_1)$ whereas Equation (44) contains q_3. Thus, after substituting Equation (45) in Equation (44), the following expression is obtained.

$$q_4 = A_1\beta_3 + A_2B_1\beta_4 + (A_2B_2 + A_3)\alpha_2 + (A_2B_3 + A_4)\alpha_1 . \qquad (46)$$

The β_1-equation gives us

$$t = \beta_1 - \frac{q_1 p_3}{\alpha_4} \mp \frac{\mu}{\alpha_4^{3/2}} \sin^{-1}\left[\frac{\mu - \alpha_4 q_1}{\sqrt{\mu^2 - \alpha_4 \alpha_3^2}}\right] .$$

Then the perturbing Hamiltonian can be expressed as

$$H_1^* = \frac{F}{m[\, t(q_1, \alpha_3, \alpha_4, \beta_1)]} \sqrt{q_3^2(q_1, \alpha, \beta_4) + q_4^2(q_1, \alpha, \beta_3, \beta_4)} . \qquad (47)$$

Thus,

$$H_1(\alpha, \beta, t) = H_1^*[\alpha, \beta, \phi(t, \alpha_3, \alpha_4, \beta_1)] \qquad (48)$$

where

$$q_1 \equiv r = \phi(t, \alpha_3, \alpha_4, \beta_1) \qquad (49)$$

is an implicit relationship defined by the β_1-equation. It follows that the perturbation equations are given by

$$\frac{d\alpha_i}{dt} = \frac{\partial H_1}{\partial \beta_i} = \frac{\partial H_1^*}{\partial \beta_i} + \frac{\partial H_1^*}{\partial r}\frac{\partial \phi}{\partial \beta_i}$$

$$(i = 1, \ldots, 4) \qquad (50)$$

$$\frac{d\beta_i}{dt} = -\frac{\partial H_1}{\partial \alpha_i} = -\frac{\partial H_1^*}{\partial \alpha_i} - \frac{\partial H_1^*}{\partial r}\frac{\partial \phi}{\partial \alpha_i} .$$

The partial derivatives of ϕ with respect to α_i and β_i can be determined by applying (ii) of the implicit function theorem (see Section I. 2A). That is, the β_1-equation defines a function

$$\psi(q_1, t, \alpha_3, \alpha_4, \beta_1) \equiv t - \beta_1 \pm \frac{\sqrt{-\alpha_3^2 + 2\mu q_1 - \alpha_4 q_1^2}}{\alpha_4}$$

$$\pm \frac{\mu}{\alpha_4^{3/2}} \sin^{-1}\left[\frac{\mu - \alpha_4 q_1}{\sqrt{\mu^2 - \alpha_4 \alpha_3^2}}\right] \equiv 0 ,$$

from which the desired partial derivations can be obtained by solving

$$\frac{\partial \psi}{\partial r}\frac{\partial \phi}{\partial \alpha_3} + \frac{\partial \psi}{\partial \alpha_3} = 0$$

$$\frac{\partial \psi}{\partial r}\frac{\partial \phi}{\partial \alpha_4} + \frac{\partial \psi}{\partial \alpha_4} = 0 \qquad (51)$$

$$\frac{\partial \psi}{\partial r}\frac{\partial \phi}{\partial \beta_1} + \frac{\partial \psi}{\partial \beta_1} = 0 .$$

The solutions of Equations (51) are:

$$\frac{\partial\phi}{\partial\alpha_3} = \frac{\alpha_3(\alpha_3^2 - \mu q_1)}{q_1(\mu^2 - \alpha_4\alpha_3^2)}$$

$$\frac{\partial\phi}{\partial\alpha_4} = \frac{D}{2\alpha_4^2(\mu^2 - \alpha_4\alpha_3^2)q_1} - \frac{3}{2}\,\frac{\mu\sqrt{-\alpha_3^2 + 2\mu q_1 - \alpha_4 q_1^2}}{q_1\,(\alpha_4)^{5/2}} \qquad (52)$$

$$\cdot\ \sin^{-1}\left(\frac{\mu - \alpha_4 q_1}{\sqrt{\mu^2 - \alpha_4\alpha_3^2}}\right)$$

$$\frac{\partial\phi}{\partial\beta_1} = -p_3 \quad ,$$

where

$$D \equiv \alpha_4(\mu^2 - \alpha_3^2\alpha_4)q_1^2 + \mu(5\alpha_3^2\alpha_4 - 6\mu^2)q_1 + \alpha_3^2(3\mu^2 - 2\alpha_3^2\alpha_4).$$

Thus the right-hand sides of Equations (50) are now well-defined functions of $\{\alpha, \beta, q_1\}$. Application of the chain rule to the left-hand sides gives

$$\frac{d\alpha_i}{dt} = \frac{d\alpha_i}{dq_1}\,\frac{dq_1}{dt} = p_3\,\frac{d\alpha_i}{dq_1}$$

$$\frac{d\beta_i}{dt} = \frac{d\beta_i}{dq_1}\,\frac{dq_1}{dt} = p_3\,\frac{d\beta_i}{dq_1} \quad ,$$

so Equations (50) can then be written as

$$\frac{d\alpha_i}{dq_1} = \frac{1}{p_3}\left[\frac{\partial H_1^*}{\partial\beta_i} + \frac{\partial H_1^*}{\partial r}\,\frac{\partial\phi}{\partial\beta_i}\right]$$

$$(i = 1, \ldots, 4) \qquad (53)$$

$$\frac{d\beta_i}{dq_1} = \frac{-1}{p_3}\left[\frac{\partial H_1^*}{\partial\alpha_i} + \frac{\partial H_1^*}{\partial r}\,\frac{\partial\phi}{\partial\alpha_i}\right] .$$

Because of the change of independent variable from t to q_1, Equations (53) are not in canonical form. If a further Hamiltonian analysis is desired, the canonical form can be regained by reworking the problem with r as the independent variable from the beginning. This presents only a slight modification to the base generating solution and canonic constants.

The expanded forms of Equations (53) are given in Appendix A.

II. 3 A Base Solution in Spherical Coordinates

The base solution which will be developed in this section is a slight modification of a base solution formed by Miner[20]. The main difference is in the use of a simple canonical transformation and the method for integration of the generating function.

Consider a spherical coordinate system (r, θ, ϕ) defined by the transformation equations

$$x = r\cos\theta\cos\phi$$
$$y = r\cos\theta\sin\phi$$
$$z = r\sin\theta\,.$$

The equations of motion in a modified spherical system are (see Fig. 2):

$$\begin{aligned}
\dot{u} &= \frac{v^2}{r^3} + \frac{w^2}{r^3\cos^2\theta} - \frac{\mu}{r^2} + \frac{F}{m}\left[\cos\theta\cos\tau\cos x^* + \sin\tau\sin\theta\right] \\
\dot{v} &= -\frac{w^2}{r^3}\sec^2\theta\tan\theta + r\frac{F}{m}\left[\sin\tau\cos\theta - \cos\tau\sin\theta\cos x^*\right] \\
\dot{w} &= r\frac{F}{m}\cos\tau\cos\theta\sin x^* \\
\dot{r} &= u
\end{aligned} \qquad (54)$$

$$\dot{\theta} = \frac{v}{r^2}$$

$$\dot{\phi} = \frac{w}{r^2 \cos^2 \theta} \qquad (54)$$

$$\dot{m} = -\sigma$$

where $x^* \equiv x - \phi$ and the mass, m, is treated as a state variable. Note that this coordinate system takes advantage of the angular momentum integral by defining $w = r^2 \dot{\phi} \cos^2 \theta$ (instead of the usual $w = r\dot{\phi} \cos \theta$). The variational Hamiltonian for this problem is

$$H^* \equiv \lambda_1 \left(\frac{v^2}{r^3} + \frac{w^2}{r^3 \cos^2 \theta} - \frac{\mu}{r^2}\right) - \lambda_2 \left(\frac{w^2}{r^2} \sec^2 \theta \tan \theta\right) + \lambda_4 u$$
$$+ \lambda_5 \frac{v}{r^2} + \lambda_6 \frac{w}{r^2 \cos^2 \theta} - \lambda_7 \sigma + \frac{F}{m} \Delta (x, \lambda) , \qquad (55)$$

where $\Delta (x, \lambda)$ is the coefficient of $\frac{F}{m}$. Since interest here is only in forming a base solution, the expression for $\Delta (x, \lambda)$ will not be developed.

Let H^* be written as

$$H^* \equiv H_o + \frac{F}{m} \Delta (x, \lambda) ,$$

where H_o is the base Hamiltonian. Note that neither $t, \phi,$ nor m appear explicitly in H_o. Thus, their conjugate momenta (i.e., $H_o, \lambda_6, \lambda_7$) are constants of the motion so the simple transformation of these vairables will be the identity transformation. To make use of the known integrals of the two-body problem, three of the state variables must be transformed to momenta and vice versa with three of the Lagrange multipliers. Thus, define:

$$p_1 = u \qquad p_2 = v \qquad p_3 = w$$

$$p_4 = \lambda_4 \qquad p_5 = \lambda_5 \qquad p_6 = \lambda_6 \qquad p_7 = \lambda_7 \tag{56}$$

Then, application of the simple transformation of Equation (I.43), i.e.,

$$S = - \sum_{i=1}^{3} x_i q_i + \sum_{i=4}^{7} x_i p_i ,$$

with

$$\lambda_i = \frac{\partial S}{\partial x_i} \qquad p_i = - \frac{\partial S}{\partial q_i} \qquad (i = 1, 2, 3)$$

$$\lambda_i = \frac{\partial S}{\partial x_i} \qquad q_i = \frac{\partial S}{\partial p_i} \qquad (i = 4, 5, 6, 7)$$

gives

$$q_1 = -\lambda_1 \qquad q_2 = -\lambda_2 \qquad q_3 = -\lambda_3$$

$$q_4 = r \qquad q_5 = \theta \qquad q_6 = \phi \qquad q_7 = m \tag{57}$$

The new base Hamiltonian is

$$H_o'(q, p) \equiv H_o[\, x(q, p), \lambda(q, p)\,] ,$$

or,

$$H_o' = -q_1 \left(\frac{p_2^2}{q_4^3} + \frac{p_3^2}{q_4^3 \cos^2 q_5} - \frac{\mu}{q_4^2} \right) + q_2 \left(\frac{p_3^2}{q_4^3} \sec^2 q_5 \tan q_5 \right)$$

$$+ p_4 p_1 + p_5 \frac{p_2}{q_4^2} + p_6 \frac{p_3}{q_4^2 \cos^2 q_5} - p_7 \sigma .$$

The H-J equation for H_o' is

$$\frac{\partial S}{\partial t} + H_o'(q, \frac{\partial S}{\partial q}) = 0 , \tag{58}$$

where $S(q, \alpha, t)$ is to be determined as a function of seven independent constants $(\alpha_1, \ldots, \alpha_7)$. Since $\frac{\partial S}{\partial t}$, $\frac{\partial S}{\partial q_6}$, and $\frac{\partial S}{\partial q_7}$ appear only once and neither t, q_6, nor q_7 appear at all in the H-J equation, it is reasonable to assume a partial separation of variables

$$S = S_1(t) + S_2(q_6) + S_3(q_7) + S^*(q_1, \ldots, q_5) . \tag{59}$$

Then substitution of Equation (59) into Equation (58) gives

$$\frac{\partial S}{\partial t} = \frac{\partial S_1}{\partial t} = \alpha_1 , \qquad \frac{\partial S}{\partial q_6} = \frac{\partial S_2}{\partial q_6} = \alpha_2 , \qquad \frac{\partial S}{\partial q_7} = \frac{\partial S_3}{\partial q_7} = \alpha_3 ,$$

so

$$S = \alpha_1 t + \alpha_2 q_6 + \alpha_3 q_7 + S^*(q_1, \ldots, q_5) . \tag{60}$$

Substitution of Equation (60) into Equation (58) and multiplication by q_4^2 gives

$$F(q_1, \ldots, q_5) \equiv q_4^2 \alpha_1 - q_1 \left[\frac{1}{q_4} \left(\frac{\partial S^*}{\partial q_2} \right)^2 + \frac{1}{q_4 \cos^2 q_5} \left(\frac{\partial S^*}{\partial q_3} \right)^2 - \mu \right] + q_2 \left[\sec^2 q_5 \tan q_5 \left(\frac{\partial S^*}{\partial q_3} \right)^2 \right] + \frac{\partial S^*}{\partial q_4} \frac{\partial S^*}{\partial q_1} q_4^2 + \frac{\partial S^*}{\partial q_5} \frac{\partial S^*}{\partial q_2} + \frac{\alpha_2}{\cos^2 q_5} \frac{\partial S^*}{\partial q_3} - \alpha_3 \sigma = 0 . \tag{61}$$

By investigating the characteristic system, three more constants can be found as follows. Since q_3 does not appear explicitly in Γ above, it follows that

$$\frac{dp_3}{d\tau} = -\frac{\partial\Gamma}{\partial q_3} = 0 \quad \rightarrow \quad p_3 = \text{constant} \equiv \alpha_4 \ ,$$

where $p_3 \equiv \dfrac{\partial S^*}{\partial q_3}$. Thus, the $\dfrac{\partial S^*}{\partial q_3}$ - terms in Equation (61) can be replaced by α_4.

Consider the characteristic equations for $p_2 \equiv \dfrac{\partial S^*}{\partial q_2}$ and q_5, i.e.,

$$\frac{dp_2}{d\tau} = -\frac{\partial\Gamma}{\partial q_2} = -\alpha_4^2 \sec^2 q_5 \tan q_5$$

$$\frac{dq_5}{d\tau} = \frac{\partial\Gamma}{\partial p_5} = p_2 \ .$$

Application of the chain rule to $\dfrac{dp_2}{d\tau}$ gives

$$\frac{dp_2}{d\tau} = \frac{dp_2}{dq_5}\frac{dq_5}{d\tau} \quad \rightarrow \quad \frac{dp_2}{dq_5} p_2 = -\alpha_4^2 \sec^2 q_5 \tan q_5 \ .$$

or,

$$p_2^2 = -2\alpha_4^2 \int \sec^2 q_5 \tan q_5 \, dq_5 + \alpha_5^2 \ ,$$

where knowledge of the two-body problem has been used as a guide in picking the constant of integration to be α_5^2 instead of α_5. Then,

$$p_2^2 = -\alpha_4^2 \sec^2 q_5 + \alpha_5^2 \ . \tag{62}$$

Consider the characteristic equations for $p_1 \equiv \frac{\partial S^*}{\partial q_1}$ and q_4, after taking advantage of Equation (62):

$$\frac{dp_1}{d\tau} = \frac{dp_1}{dq_4}\frac{dq_4}{d\tau} = \frac{(\alpha_5^2 - \alpha_4^2 \sec^2 q_5)}{q_4} + \frac{\alpha_4^2}{q_4 \cos^2 q_5} - \mu$$

$$\frac{dq_4}{d\tau} = q_4^2 p_1$$

or

$$p_1 \frac{dp_1}{dq_4} = \frac{(\alpha_5^2 - \alpha_4^2 \sec^2 q_5)}{q_4^3} + \frac{\alpha_4^2}{q_4^3 \cos^2 q_5} - \frac{\mu}{q_4^2}$$

or

$$\frac{1}{2}\frac{d}{dq_4}(p_1^2) = \frac{\alpha_5^2}{q_4^3} - \frac{\mu}{q_4^2} .$$

Upon integration, the energy integral is obtained:

$$p_1^2 = \frac{2\mu}{q_4} - \frac{\alpha_5^2}{q_4^2} - \alpha_6 . \tag{63}$$

Six constants of the motion have now been obtained. Jacobi's theorem requires seven for a complete solution to the base solution. The final constant can be obtained by substituting Equations (62) and (63) into Equation (61) and noting that another separation of variables is possible, i.e., Equation (61) can be written as

$$q_4^2(\alpha_1 - \alpha_3\sigma) - q_1\left(\frac{\alpha_5^2}{q_4} - \mu\right) + q_4^2 \frac{\partial S^*}{\partial q_4} \frac{\partial S^*}{\partial q_1}$$

$$= -\frac{\partial S^*}{\partial q_5} \frac{\partial S^*}{\partial q_2} - \alpha_4^2 q_2 \sec^2 q_5 \tan q_5 - \alpha_2\alpha_4 \sec^2 q_5 \, . \qquad (64)$$

Thus, assume a solution of the form:

$$S = \alpha_1 t + \alpha_2 q_6 + \alpha_3 q_7 + \alpha_4 q_3 + S'(q_1, q_4) + S''(q_2, q_5). \qquad (65)$$

Again making use of the two-body problem, let the new constant be denoted by $\alpha_7\alpha_5$. Then,

$$\alpha_7\alpha_5 = q_4^2(\alpha_1 - \alpha_3\sigma) - q_1\left(\frac{\alpha_5^2}{q_4} - \mu\right) + q_4^2 \frac{\partial S'}{\partial q_4} \frac{\partial S'}{\partial q_1} \qquad (66)$$

and,

$$\alpha_7\alpha_5 = -\frac{\partial S''}{\partial q_5} \frac{\partial S''}{\partial q_2} - \alpha_4^2 q_2 \sec^2 q_5 \tan q_5 - \alpha_2\alpha_4 \sec^2 q_5 \, ,$$

where each side of Equation (54) must be constant since the left-hand side is only a function of q_1 and q_4; and the right-hand side is only a function of q_2 and q_5.

The functions $S'(q_1, q_4)$ and $S''(q_2, q_5)$ must be determined in order to define the generating function S. To find S' and S'', two Pfaff differential equations must be integrated, i.e.,

$$dS' = \frac{\partial S'}{\partial q_1} dq_1 + \frac{\partial S'}{\partial q_4} dq_4$$

and, (67)

$$dS'' = \frac{\partial S''}{\partial q_2} dq_2 + \frac{\partial S''}{\partial q_5} dq_5 \quad ,$$

where

$$\frac{\partial S'}{\partial q_1} = \frac{\partial S'}{\partial q_1}(q_4) = \pm \frac{1}{q_4}\sqrt{-\alpha_6 q_4^2 + 2\mu q_4 - \alpha_5^2}$$

$$\frac{\partial S'}{\partial q_4} = \pm \frac{1}{q_4\sqrt{-\alpha_6 q_4^2 + 2\mu q_4 - \alpha_5^2}}\left\{\alpha_5\alpha_7 - q_4^2(\alpha_1 - \alpha_3\sigma) + q_1\left(\frac{\alpha_5^2}{q_4} - \mu\right)\right\}$$

$$\frac{\partial S''}{\partial q_2} = \frac{\partial S''}{\partial q_2}(q_5) = \pm\sqrt{\alpha_5^2 - \alpha_4^2 \sec^2 q_5} \tag{68}$$

$$\frac{\partial S''}{\partial q_5} = \pm\frac{1}{\sqrt{\alpha_5^2 - \alpha_4^2\sec^2 q_5}}\left\{-\alpha_5\alpha_7 - \alpha_2\alpha_4 \sec^2 q_5 - \alpha_4^2 q_2 \sec^2 q_5 \tan q_5\right\}.$$

Thus, Equations (67) are of the functional form

$$dS' = R_1(q_4)\,dq_1 + R_2(q_1, q_4)\,dq_4$$

$$dS'' = T_1(q_5)\,dq_2 + T_2(q_2, q_5)\,dq_5 .$$

As shown in Section I.2B, since $\frac{\partial S'}{\partial q_1}$ only depends upon q_4 and $\frac{\partial S''}{\partial q_2}$ only depends upon q_5, then

$$S' = R_1(q_4)\,q_1 + S_o'(q_4)$$

$$S'' = T_1(q_5)\,q_2 + S_o''(q_5) .$$

where $S'_o(q_4)$ and $S''_o(q_4)$ can be obtained by integrating $\frac{\partial S'}{\partial q_4}$ without the q_1-term and $\frac{\partial S''}{\partial q_5}$ without the q_2-term, respectively. Then,

$$S'_o(q_4) = \frac{(\alpha_1 - \alpha_3\sigma)q_4 p_1}{\alpha_6} \pm \left\{ \frac{(\alpha_1 - \alpha_3\sigma)\mu}{\alpha_6^{3/2}} \sin^{-1}\left(\frac{\mu - \alpha_6 q_4}{\sqrt{\mu^2 - \alpha_6\alpha_5^2}}\right) + \alpha_7 \sin^{-1}\left(\frac{\mu q_4 - \alpha_5^2}{q_4\sqrt{\mu^2 - \alpha_6\alpha_5^2}}\right) \right\}$$

and, (69)

$$S''_o(q_5) = \mp \left\{ \alpha_7 \sin^{-1}\left(\frac{\alpha_5 \sin q_5}{\sqrt{\alpha_5^2 - \alpha_4^2}}\right) + \alpha_2 \sin^{-1}\left(\frac{\alpha_4 \tan q_5}{\sqrt{\alpha_5^2 - \alpha_4^2}}\right) \right\}.$$

As in the planar case, Equations (69) are only valid for elliptical motion.

Substitution of Equations (68) into Equation (65) leads to the generating function for the base problem:

$$\begin{aligned} S = {} & \alpha_1 t + \alpha_2 q_6 + \alpha_3 q_7 + \alpha_4 q_3 \pm \frac{1}{q_4}\sqrt{-\alpha_6 q_4^2 + 2\mu q_4 - \alpha_5^2}\; q_1 \\ & \pm \sqrt{\alpha_5^2 - \alpha_4^2 \sec^2 q_5}\; q_2 \pm \alpha_7 \sin^{-1}\left(\frac{\mu q_4 - \alpha_5^2}{q_4\sqrt{\mu^2 - \alpha_6\alpha_5^2}}\right) \\ & \pm \frac{(\alpha_1 - \alpha_3\sigma)}{\alpha_6}\sqrt{-\alpha_6 q_4^2 + 2\mu q_4 - \alpha_5^2} \\ & \pm \frac{\mu(\alpha_1 - \alpha_3\sigma)}{\alpha_6^{3/2}} \sin^{-1}\left(\frac{\mu - \alpha_6 q_4}{\sqrt{\mu^2 - \alpha_6\alpha_5^2}}\right) \\ & \pm \alpha_7 \sin^{-1}\left(\frac{\alpha_5 \sin q_5}{\sqrt{\alpha_5^2 - \alpha_4^2}}\right) \pm \alpha_2 \sin^{-1}\left(\frac{\alpha_4 \tan q_5}{\sqrt{\alpha_5^2 - \alpha_4^2}}\right). \end{aligned} \qquad (70)$$

Then, Jacobi's theorem can be applied to give the remaining seven constants of the motion, i.e., $\beta_i = \frac{\partial S}{\partial \alpha_i}$ for $i = 1, \ldots, 7$. Thus, the total set of constants of the motion are:

$$\alpha_1 = q_1 \left(\frac{p_2^2}{q_4^2} + \frac{p_3^2}{q_4^2 \cos^2 q_5} - \frac{\mu}{q_4^2} \right) - q_2 \left(\frac{p_3^2}{q_4^2} \sec^2 q_5 \tan q_5 \right)$$

$$- p_4 p_1 - p_5 \frac{p_2}{q_4^2} - p_6 \frac{p_3}{q_4^2 \cos^2 q_5} + p_7 \sigma$$

$$\alpha_2 = p_6$$

$$\alpha_3 = p_7$$

$$\alpha_4 = p_3$$

$$\alpha_5 = \pm \sqrt{p_2^2 + p_3^2 \sec^2 q_5} \tag{71}$$

$$\alpha_6 = \frac{2\mu}{q_4} - \frac{1}{q_4^2} (p_2^2 + p_3^2 \sec^2 q_5) - p_1^2$$

$$\alpha_7 = \pm \frac{1}{\sqrt{p_2^2 + p_3^2 \sec^2 q_5}} \left[q_4^2(\alpha_1 - p_7 \sigma) + q_4^2 p_1 p_4 \right.$$

$$\left. - \frac{q_1}{q_4} (p_2^2 + p_3^2 \sec^2 q_5 - \mu q_4) \right]$$

$$\beta_1 = t \pm \frac{1}{\alpha_6} \sqrt{-\alpha_6 q_4^2 + 2\mu q_4 - \alpha_5^2} \pm \frac{\mu}{\alpha_6^{3/2}} \sin^{-1} \left(\frac{\mu - \alpha_6 q_4}{\sqrt{\mu^2 - \alpha_6 \alpha_5^2}} \right)$$

$$\beta_2 = q_6 \mp \sin^{-1} \left(\frac{\alpha_4 \tan q_5}{\sqrt{\alpha_5^2 - \alpha_4^2}} \right)$$

$$\beta_3 = q_7 \mp \frac{\sigma}{\alpha_6} \sqrt{-\alpha_6 q_4^2 + 2\mu q_4 - \alpha_5^2} \mp \frac{\sigma\mu}{\alpha_6^{3/2}} \sin^{-1} \left(\frac{\mu - \alpha_6 q_4}{\sqrt{\mu^2 - \alpha_6 \alpha_5^2}} \right)$$

$$\beta_4 = q_3 - \frac{\alpha_4 q_2 \sec^2 q_5}{p_2} - \frac{\alpha_5 \tan q_5 (\alpha_2 \alpha_5 + \alpha_4 \alpha_7)}{p_2 (\alpha_5^2 - \alpha_4^2)}$$

$$\beta_5 = \frac{-\alpha_5 q_1}{q_4^2 p_1} + \frac{\alpha_5 q_2}{p_2} + \frac{\alpha_7}{q_4 p_1} \left(\frac{\alpha_6 \alpha_5^2 + \mu q_4 \alpha_6 - 2\mu^2}{\mu^2 - \alpha_6 \alpha_5^2} \right)$$

$$- \frac{\alpha_5 (\alpha_1 - \alpha_3 \sigma)}{\alpha_6 q_4 p_1} + \frac{\mu \alpha_5 (\alpha_1 - \alpha_3 \sigma)(\mu - \alpha_6 q_4)}{\alpha_6 q_4 p_1 (\mu^2 - \alpha_6 \alpha_5^2)}$$

$$+ \frac{(\alpha_7 \alpha_4^2 + \alpha_2 \alpha_5 \alpha_4)}{p_2 (\alpha_5^2 - \alpha_4^2)} \tan q_5$$

$$\beta_6 = - \frac{q_1}{2p_1} + \frac{\alpha_5 \alpha_7 (\mu q_4 - \alpha_5^2)}{2 q_4 p_1 (\mu^2 - \alpha_6 \alpha_5^2)}$$

$$\mp \frac{3}{2} \frac{(\alpha_1 - \alpha_3 \sigma)\mu}{\alpha_6^{5/2}} \sin^{-1} \left(\frac{\mu - \alpha_6 q_4}{\sqrt{\mu^2 - \alpha_6 \alpha_5^2}} \right)$$

$$+ \frac{(\alpha_1 - \alpha_3 \sigma)[(\mu^2 \alpha_6 - \alpha_5^2 \alpha_6^2) q_4^2 + (5\alpha_6 \alpha_5^2 \mu - 6\mu^3) q_4 + (3\mu^2 \alpha_5^2 - 2\alpha_6 \alpha_5^4)]}{2\alpha_6^2 (\mu^2 - \alpha_6 \alpha_5^2) q_4 p_1}$$

$$\beta_7 = \pm \sin^{-1} \left(\frac{\mu q_4 - \alpha_5^2}{q_4 \sqrt{\mu^2 - \alpha_6 \alpha_5^2}} \right) \mp \sin^{-1} \left(\frac{\alpha_5 \sin q_5}{\sqrt{\alpha_5^2 - \alpha_4^2}} \right) .$$

The set $\{\alpha, \beta\}$ represents the solution to the coast-arc problem in spherical coordinates. As with the planar problem, Equations (71) possess singularities at $\dot{r} \equiv p_1 = 0$ and energy = 0. Thus, this solution is also restricted to elliptical trajectories on which $\dot{r} > 0$ or $\dot{r} < 0$.

PART III

CONCLUSIONS AND RECOMMENDATIONS

III. 1 Summary

In the discussion presented in the previous sections, the application of canonical transformation theory to the problem of obtaining approximate analytic solutions for the motion of a continuously-thrusting space vehicle has been described. The pertinent aspects of the theory are rigorously discussed in Part I. In Part II, the application of the theory to the optimal trajectory problem is described, and a base solution for the planar problem, as formulated in polar coordinates, is obtained and the time rates of change of the base canonic constants for the planar problem due to the effects of the engine thrust are developed. Finally, a modification of the spherical coordinate system base solution given by Miner[20], is presented.

III. 2 Conclusions and Recommendations

Based on the results obtained in the previous sections, it appears that canonical perturbation theory can be used to obtain approximate analytic solutions to the equations which govern the optimal motion of a continuously-thrusting space vehicle. The constants defined by the base solution can be used to describe the coast or zero-thrust portions of the trajectory. The equations which govern the time rates of change of these constants under the effects of the thrust can be used for numerical studies of the vehicle motion.

However, the base solutions presented contain singularities for the cases where the radial velocity and the energy are equal to zero. The singularities restrict the base solution to elliptical orbits, only. As a consequence, the base solutions are not applicable in their entirety to such missions as escape trajectories, Earth-Mars transfer trajectories, and near-orbital transfers.

It is recommended that future investigations be directed towards obtaining a more general base solution which is valid for both elliptical and circular orbits. Consideration should be given also, to obtaining solutions which are valid for both parabolic and hyperbolic trajectory conditions. Finally, the Hamilton-Jacobi equation for the remaining perturbing Hamiltonian should be examined in an effort to incorporate the effects of the thrusting engine in the analytical solution.

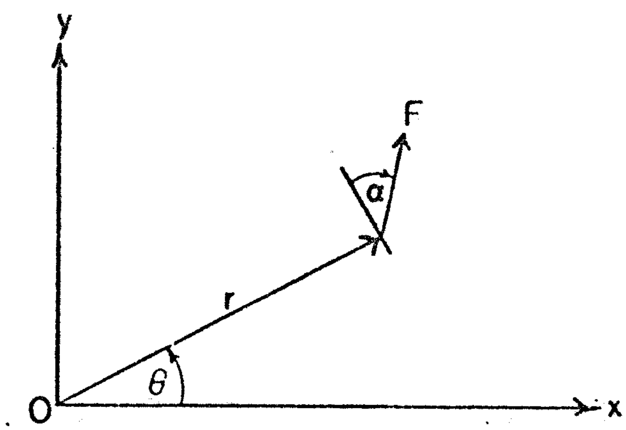

FIG. 1. Planar Representation

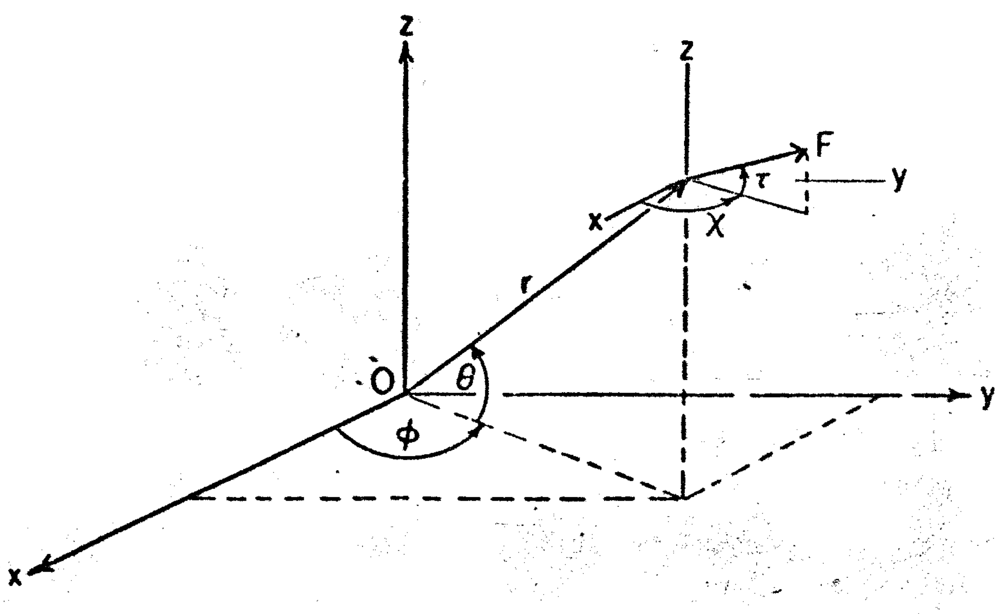

FIG. 2. Spherical Representation

APPENDIX A

In Section II. 2B, a method for the generation of the rates of change of the base canonic constants with respect to radius was presented. The results of such an analysis for the Hamiltonian of Equation II-29 are given below.

$$\frac{d\alpha_1}{dr} = -\frac{F/m}{u\sqrt{\lambda_3^2+\lambda_4^2}}\left(\lambda_3 \frac{\partial\alpha_1}{\partial u} + \lambda_4 \frac{\partial\alpha_1}{\partial v}\right)$$

$$\frac{d\alpha_2}{dr} = 0 \longrightarrow \alpha_2 \equiv \text{constant}$$

$$\frac{d\alpha_3}{dr} = -\frac{F/m}{u\sqrt{\lambda_3^2+\lambda_4^2}}\left(r\lambda_4\right)$$

$$\frac{d\alpha_4}{dr} = \frac{2\,F/m}{u\sqrt{\lambda_3^2+\lambda_4^2}}\left(\lambda_3 u + \lambda_4 \frac{\alpha_3}{r}\right)$$

$$\frac{d\beta_1}{dr} = -\frac{F/m}{u\sqrt{\lambda_3^2+\lambda_4^2}}\left(\lambda_3 \frac{\partial\beta_1}{\partial u} + \lambda_4 \frac{\partial\beta_1}{\partial v}\right)$$

$$\frac{d\beta_2}{dr} = -\frac{F/m}{u\sqrt{\lambda_3^2+\lambda_4^2}}\left(\lambda_3 \frac{\partial\beta_2}{\partial u} + \lambda_4 \frac{\partial\beta_2}{\partial v}\right)$$

$$\frac{d\beta_3}{dr} = -\frac{F/m}{u\sqrt{\lambda_3^2+\lambda_4^2}}\left(\lambda_3 \frac{\partial\beta_3}{\partial u} + \lambda_4 \frac{\partial\beta_3}{\partial v}\right)$$

$$\frac{d\beta_4}{dr} = -\frac{F/m}{u\sqrt{\lambda_3^2+\lambda_4^2}}\left(\lambda_3 \frac{\partial\beta_4}{\partial u} + \lambda_4 \frac{\partial\beta_4}{\partial v}\right),$$

where

$$\frac{\partial\alpha_1}{\partial u} = -\frac{1}{u}\left[\alpha_1 + \frac{\alpha_2\alpha_3}{r^2} + \lambda_3\left(\frac{\alpha_3^2}{r^3} - \frac{\mu}{r^2}\right) + \lambda_4 \frac{\alpha_3 u}{r^2}\right] + \frac{\alpha_3\lambda_4}{r^2}$$

$$\frac{\partial \alpha_1}{\partial v} = -\frac{\alpha_2}{r} - \lambda_3 \frac{2\alpha_3}{r} + \lambda_4 \frac{u}{r}$$

$$\frac{\partial \beta_1}{\partial u} = \frac{r}{\alpha_4}\left[1 + \frac{2u^2}{\alpha_4}\right] + \frac{\mu}{\alpha_4^2}\left[2 - \frac{\alpha_3^2}{r}\,\frac{\mu - \alpha_4 r}{\mu^2 - \alpha_4\alpha_3^2}\right] \pm \frac{3\mu u}{\alpha_4^{5/2}} \sin^{-1}\left[\frac{\mu - \alpha_4 r}{\sqrt{\mu^2 - \alpha_4\alpha_3^2}}\right]$$

$$\frac{\partial \beta_1}{\partial v} = \frac{\alpha_3}{u\alpha_4^2}\left[2u^2 + \frac{2\mu}{r} + \frac{(\mu - \alpha_4 r)(\alpha_4 - \frac{\alpha_3}{r^2})}{\mu^2 - \alpha_4\alpha_3^2}\right] \pm \frac{3\mu\alpha_3}{r\alpha_4^{5/2}} \sin^{-1}\left[\frac{\mu - \alpha_4 r}{\sqrt{\mu^2 - \alpha_4\alpha_3^2}}\right]$$

$$\frac{\partial \beta_2}{\partial u} = \frac{\alpha_3(\mu - \frac{\alpha_3^2}{r})}{\mu^2 - \alpha_4\alpha_3^2}$$

$$\frac{\partial \beta_2}{\partial v} = \frac{1}{u}\left[2 - r(\alpha_4 - \frac{\alpha_3^2}{r^2})\,\frac{(\mu - \frac{\alpha_3^2}{r})}{\mu^2 - \alpha_4\alpha_3^2}\right]$$

$$\frac{\partial \beta_3}{\partial u} = \frac{\alpha_2}{r(\mu^2 - \alpha_4\alpha_3^2)}\left[(\mu - \frac{\alpha_3^2}{r})(\frac{2\mu}{r} - \alpha_4) - \frac{2\mu\alpha_3^2(\mu - \alpha_4 r)}{\mu^2 - \alpha_4\alpha_3^2}\right] - \frac{\lambda_3\alpha_3}{ru^2}$$
$$+ \frac{\partial \alpha_1}{\partial u}\,\frac{\alpha_3}{ru}\left[\frac{\alpha_3^2 - \mu r}{\mu^2 - \alpha_4\alpha_3^2}\right] - \frac{\alpha_1\alpha_3(\alpha_3^2 - \mu r)}{r(\mu^2 - \alpha_4\alpha_3^2)}\left[\frac{1}{u^2} + \frac{2\alpha_3}{\mu^2 - \alpha_4\alpha_3^2}\right]$$

$$\frac{\partial \beta_3}{\partial v} = \frac{2\mu\alpha_2\alpha_3}{ru(\mu^2 - \alpha_4\alpha_3^2)}\left[1 + \frac{\mu - \alpha_4 r}{r}\cdot\frac{\alpha_4 - \frac{\alpha_3^2}{r^2}}{\mu^2 - \alpha_4\alpha_3^2}\right] + \frac{\lambda_3}{ru} + \frac{\partial \alpha_1}{\partial v}\cdot\frac{\alpha_3(\alpha_3^2 - \mu r)}{ru(\mu^2 - \alpha_4\alpha_3^2)}$$
$$+ \frac{\alpha_1}{u(\mu^2 - \alpha_4\alpha_3^2)}\left[3\alpha_3^2 - \mu r + 2\alpha_3^2(\alpha_3^2 - \mu r)\left(\frac{\alpha_4 - \frac{\alpha_3^2}{r^2}}{\mu^2 - \alpha_4\alpha_3^2}\right)\right]$$

$$\frac{\partial \beta_4}{\partial u} = -\frac{\alpha_2\alpha_3(\alpha_3^2 - \mu r)}{r(\mu^2 - \alpha_4\alpha_3^2)}\left[\frac{1}{2u^2} + \frac{\alpha_3^2}{\mu^2 - \alpha_4\alpha_3^2}\right] - \frac{\lambda_3}{2u^2} + \frac{\partial \alpha_1}{\partial u} D + \alpha_1 \frac{\partial D}{\partial u}$$

$$\frac{\partial\beta_4}{\partial v} = \frac{\alpha_2}{u(\mu^2 - \alpha_4\alpha_3^2)}\left[\frac{3\alpha_3^2 - \mu r}{2} + \alpha_3^2(\alpha_3^2 - \mu r)\,\frac{\alpha_4 - \frac{\alpha_3^2}{r^2}}{\mu^2 - \alpha_4\alpha_3^2}\right] + \frac{\partial\alpha_1}{\partial v} D + \alpha_1 \frac{\partial D}{\partial v}$$

$$D = \frac{\mu^2\left(\frac{\alpha_3^2}{r} - \mu\right)}{2u\alpha_4^2(\mu^2 - \alpha_4\alpha_3^2)} - \frac{r\left(u^2 - \frac{\alpha_3^2}{r^2} + \frac{3\mu}{r}\right)}{2u\alpha_4^2} \mp \frac{3}{2}\,\frac{\mu}{\alpha_4^{5/2}}\,\sin^{-1}\left[\frac{\mu - \alpha_4 r}{\sqrt{\mu^2 - \alpha_4\alpha_3^2}}\right]$$

$$\frac{\partial D}{\partial u} = \frac{\mu^2\left(\frac{\alpha_3^2}{r} - \mu\right)}{\alpha_4^2(\mu^2 - \alpha_4\alpha_3^2)}\left[-\frac{1}{2u^2} + \frac{2}{\alpha_4} + \frac{\alpha_3^2}{\mu^2 - \alpha_4\alpha_3^2}\right] +$$

$$+ \frac{r}{\alpha_4^2}\left(-\frac{2}{\alpha_4} + \frac{1}{2u^2}\right)\left(u^2 - \frac{\alpha_3^2}{r^2} + \frac{3\mu}{r}\right)$$

$$- \frac{1}{\alpha_4^2}\left[1 + \frac{3\mu}{\alpha_4} - \frac{3\mu\alpha_3^2}{2r\alpha_4}\left(\frac{\mu - \alpha_4 r}{\mu^2 - \alpha_4\alpha_3^2}\right)\right] \mp \frac{15}{2}\,\frac{\mu}{\alpha_4^{7/2}}\,\sin^{-1}\left[\frac{\mu - \alpha_4 r}{\sqrt{\mu^2 - \alpha_4\alpha_3^2}}\right]$$

$$\frac{\partial D}{\partial v} = \frac{\mu^2\alpha_3}{u\alpha_4^2(\mu^2 - \alpha_4\alpha_3^2)}\left[1 + (\alpha_3^2 - \mu r)\left(\frac{2}{r^2\alpha_4} + \frac{\alpha_4 - \frac{\alpha_3^2}{r^2}}{\mu^2 - \alpha_4\alpha_3^2}\right)\right]$$

$$+ \frac{\alpha_3}{u\alpha_4^3}\left[-3u^2 + \frac{\alpha_3^2}{r^2} - \frac{7\mu}{r} - \frac{3\mu}{2}\,\frac{(\mu - \alpha_4 r)\left(\alpha_4 - \frac{\alpha_3^2}{r^2}\right)}{(\mu^2 - \alpha_4\alpha_3^2)}\right]$$

$$\mp \frac{15}{2}\,\frac{\mu\alpha_3}{r\alpha_4^{7/2}}\,\sin^{-1}\left[\frac{\mu - \alpha_4 r}{\sqrt{\mu^2 - \alpha_4\alpha_3^2}}\right]$$

The expressions for $p_3 \equiv u(r, \alpha)$, $\lambda_3(r, \alpha, \beta)$, and $\lambda_4(r, \alpha, \beta)$ are defined by Equations (II-43), (II-53), and (II-52'), respectively.

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